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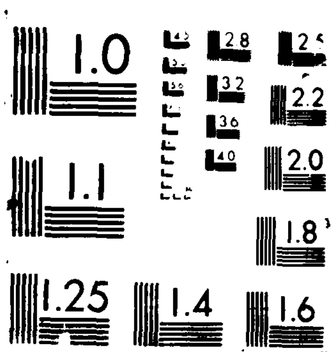
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The Hamiltonian Structure of Nonlinear Elasticity:

The Convective Representation of Solids, Rods, and Plates

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Contents

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§1. *Introduction*

It is our belief that a thorough understanding of the mathematical underpinnings of elasticity is crucial to its analytical and numerical implementation. For example, in the analysis of rotating structures, if one attempts to couple geometrically inexact models, obtained by linearization or other approximations to rotating rigid bodies, one can easily get serious artificial "softening" effects that can significantly alter numerical results; see Simo and Vu-Quoc [1986c] for a discussion (compare equations (3) and (5) of that paper). In this paper, we consider geometrically exact models, such as the Kirchhoff-Love-Reissner-Antman model for rods and its counterpart for plates and shells. These models take into account shear and torsion as well as the usual bending effects in traditional rod and plate models. Our purpose is to systematically develop the Hamiltonian structure for the dynamics of these models in the convective representation. The convective representation is chosen for its computational convenience and for our planned coupling of these models to the dynamics of rigid body motion, as in Krishnaprasad and Marsden [1986].

One of the topics that is of importance in the foundations of elasticity is a geometric formulation of the equations in Hamiltonian form. This form is extremely useful in the dynamical analysis of systems; for example in the study of nonlinear stability (see Holm, Marsden, Ratiu, and Weinstein [1985], Krishnaprasad [1985], and Lewis, Marsden, and Ratiu [1986a]), in bifurcation theory (see Golubitsky and Stewart [1986] and Lewis, Marsden and Ratiu [1986b]) and in the study of chaotic solutions (see Holmes and Marsden [1983] and Guckenheimer and Holmes [1983]). Our own motivation is to provide additional insight for work on rotating structures using geometrically exact models (see Krishnaprasad and Marsden [1986] and Krishnaprasad, Marsden, and Simo [1987]). These motivations are of course in addition to the fact that these Hamiltonian structures are of intrinsic interest in themselves for the mathematical foundations of elasticity theory.

The Hamiltonian structure for the material (or Lagrangian) representation of elasticity is given in terms of canonically conjugate variables - namely the placement field and its conjugate momentum, the momentum density. This is a fairly standard result that is given in for example, Marsden and Hughes [1983, Chapter 5]. The relation beween this and other structures in spatial and body representations is an important development that goes back to Arnold [1966] and was developed by Marsden and Weinstein [1974, 1982], and others. A non canonical Hamiltonian structure for elasticity that is partially a spatial representation is given in Holm and Kuperschmit [1983], and a Hamiltonian structure for isotropic elasticity in spatial representation is given in Marsden, Ratiu, and Weinstein [1984a,b]. We concentrate in this paper on the *convective representation* and also develop a Hamiltonian formalism for rods and plates. This is done for

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several reasons. First of all, no assumption of isotropy is required. Second, the convective representation is convenient for the analysis of coupled rigid body - beam and plate structures. Finally, the convective representation is very convenient for numerical computations for geometrically exact models (see Simo and Vu-Quoc [1986a,b]). A geometric setting that is useful for understanding the general relation between the material, inverse material, spatial, and convective representations is given in Holm, Marsden, and Ratiu [1986].

As we have mentioned, the noncanonical brackets found in this paper are obtained by the general methods of reduction from the canonical structure in material representation, as in Arnold [1966] and Marsden and Weinstein [1982]. When these procedures are done for fixed boundary problems, one obtains Lie-Poisson brackets associated with the dual of a Lie algebra of a semi-direct product. (See Marsden, Weinstein, Ratiu, Schmit, and Spencer [1983] for a general introduction to this geometric theory.) These sorts of brackets appear for example in the equations for compressible fluids and MHD (see Marsden [1982], Holm and Kuperschmit [1983] and Marsden, Ratiu, and Weinstein [1984a,b]). The geometric reason for this is that when one starts with a configuration space that is a group G, and one reduces the phase space T^*G by a subgroup G_a that is the isotropy subgroup for a representation of G on a vector space V, then the resulting space is essentially the dual of the Lie algebra of the semi-direct product $G \circledS V$. (This result, due to Ratiu, Guillemin and Sternberg, is proved in a sharpened version in Marsden, Ratiu, and Weinstein [1984a] to which we refer for the original references). When free boundaries are present, however, the brackets are only partially of the Lie-Poisson type. The geometric setting for these is the "gauged Lie-Poisson" context of Montgomery, Marsden, and Ratiu [1984]. This was applied to free boundary fluid problems in Lewis, Marsden, Montgomery, and Ratiu [1986]. In this paper, we shall not require the fairly sophisticated context of the gauged Lie-Poisson structures, but rather we shall obtain the results by a direct calculation. However, we do note that when no boundaries are present, the Poisson brackets we get for three dimensional elasticity do reduce to Lie-Poisson brackets for a semidirect product. For rods and plates, the brackets also reduce to the Lie-Poisson type in the cases that the configuration space reduces to a group; for example this happens for torsional motion of a rod.

The geometric point of view adopted in this paper has proven particularly useful in the numerical solution of initial boundary value problems. For the geometrically exact rod model, for instance, *exact update procedures* for the configuration, stress resultants and stress couples can be developed by employing discrete algorithmic counterparts of the exponential map and parallel transport (see Simo & Vu-Quoc [1986a,b]). This results in algorithms that *exactly preserve* the fundamental physical requirement of material frame indifference. Similarly, for three dimensional nonlinear viscoelastic solids, by exploiting the convective representation, one can develop uncondionally stable and second order accurate algorithms which exactly preserve covariance of the

continuum formulation (see Simo [1986]). Thus, these algorithms go beyond the notion of incremental objectivity, as proposed by Hughes and Winget [1980]. Finally, we believe that the Hamiltonian structures developed in this paper will play a central role in future developement, design, and stability analysis of time-stepping integration algorithms for nonlinear elastodynamics which ensure not only conservation of energy, (as in Chorin et. al. [1978] or Hughes, Liu & Caughy [1980]), but exactly preserve other fundamental integrals of motion such as global angular momentum.

Acknowledgements. We thank David Fox, Darryl Holm, Tom Hughes, Debbie Lewis, Tudor Ratiu, and Loc Vu-Quoc for their comments and joint discussions.

§2. *Summary of Three Dimensional Elasticity*

We summarize the notation to be used in the description of three dimensional elastodynamics, following to a large extent the useage of Marsden and Hughes [1983].

The Configuration Space

Let $(\mathcal{B}, G)$ and $(\mathcal{S}, g)$ be two smooth Riemannian manifolds carrying metrics G and g respectively. Typically we have $\mathcal{B} \subset \mathcal{S}$ and in fact $\mathcal{S} = \mathbb{R}^3$, Euclidean three-space with the standard Euclidean metric. One refers to $\mathcal{B}$ as the ***reference configuration*** with points denoted by $X \in \mathcal{B}$, and one speaks of $\mathcal{S}$ as the ***ambient space*** in which the body $\mathcal{B}$ moves. Points in $\mathcal{S}$ are denoted $x \in \mathcal{S}$. We shall consider coordinate charts $\hat{X}^A : \mathcal{B} \longrightarrow \mathbb{R}$ and $\hat{x}^a : \mathcal{S} \longrightarrow \mathbb{R}$ so that the local coordinates of the points X and x are denoted

$$X^A = \hat{X}^A(X), \quad \text{for} \quad X \in \mathcal{B} \quad \text{and} \quad x^a = \hat{x}^a(x), \quad \text{for} \quad x \in \mathcal{S} \tag{2.1}$$

The configuration space $\mathcal{C}$ is the set of (orientation preserving) embeddings $\varphi : \mathcal{B} \longrightarrow \mathcal{S}$; we write

$$\mathcal{C} = \mathrm{Emb}(\mathcal{B}, \mathcal{S}) \tag{2.2}$$

and call the set $\varphi(\mathcal{B})$ the ***current confuguration***. It is known that, when suitably tcpologized, $\mathcal{C}$ is a smooth infinite dimensional manifold (see Abraham, Marsden, and Ratiu [1983], Ebin and Marsden [1970] and references therein).

The tangent space to $\mathcal{C}$ at a configuration φ is given by

$$T_\varphi \mathcal{C} = \{ V : T\mathcal{B} \longrightarrow T\mathcal{S} \mid V(X) \in T_{\varphi(X)}\mathcal{S} \text{ for all } X \in \mathcal{B} \} \tag{2.3}$$

Kinematics

A ***motion*** is a curve of configurations; we let φ_t be the configuration at time t and write $\varphi_t(X) = \varphi(X,t)$. Given a motion φ_t, we define the following quantities:

(i) material velocity : $V_t \in T_{\varphi_t}\mathcal{C}$ given by

$$V_t(X) := \frac{\partial}{\partial t} \varphi(X, t) \tag{2.4a}$$

(ii) spatial velocity : $v_t \in \mathfrak{X}(\varphi_t(\mathcal{B}))$ [the space of vector fields on $\varphi_t(\mathcal{B})$] is defined by

$$v_t = V_t \circ \varphi_t^{-1} \tag{2.4b}$$

(iii) convective velocity : $\mathcal{V}_t \in \mathfrak{X}(\mathcal{B})$ is defined by

$$\mathcal{V}_t = \varphi_t^*(v_t) := T\varphi_t^{-1} \circ v_t \circ \varphi_t = T\varphi_t^{-1} \circ V_t \tag{2.4c}$$

The *deformation gradient,* denoted F_t is defined to be the tangent map of φ_t; we write $F_t = T\varphi_t$. In coordinates,

$$F^a{}_A = \frac{\partial \varphi^a}{\partial X^A}.$$

Proposition 2.1. *The convected velocity is minus the spatial velocity of the inverse motion* $\varphi_t^{-1} : \mathcal{S} \longrightarrow \mathcal{B}$ *; i.e.,*

$$\mathcal{V}_t = -\frac{\partial \varphi_t^{-1}}{\partial t} \circ \varphi_t \tag{2.5}$$

Proof. Applying the chain rule to the identity $X = \varphi^{-1}(\varphi(X, t), t)$ gives

$$\frac{\partial \varphi_t^{-1}}{\partial t} \circ \varphi_t + T\varphi_t^{-1} \circ V_t = 0$$

and so the result follows by noting that $\mathcal{V}_t = T\varphi_t^{-1} \circ v_t \circ \varphi_t$. ■

The Metric and Convected Metric Tensors

The convected metric tensor is defined by the pull-back relation:

$$C_t = \varphi_t^*(g). \tag{2.6a}$$

One refers to $\mathbf{C}_t$ as the *right Cauchy-Green* tensor. In coordinates,

$$C_{AB} = F^a{}_A F^b{}_B g_{ab} \circ \varphi. \tag{2.6b}$$

Associated with the convected metric $\mathbf{C}_t$ we define a symmetric ***Levi-Cevita connection*** ∇ to be the connection of the metric $\mathbf{C}_t$; i.e. by the standard relation for the Christoffel symbols:

$$(\nabla_{\mathcal{V}}\mathcal{W})^B = \mathcal{V}^A \mathcal{W}^B{}_{,A} + \Gamma_{AD}{}^B \mathcal{V}^A \mathcal{W}^D \tag{2.7a}$$

where

$$\Gamma_{AD}{}^B = C^{BE}\Gamma_{AD,E} \tag{2.7b}$$

and

$$\Gamma_{AB,C} = \frac{1}{2}\left[\frac{\partial C_{AC}}{\partial X^B} + \frac{\partial C_{BC}}{\partial X^A} - \frac{\partial C_{AB}}{\partial X^C}\right] \tag{2.7c}$$

For the case of Euclidean space one has the formula

$$\Gamma_{AD}{}^B = \frac{\partial^2\varphi^a}{\partial X^A \partial X^D}(F^{-1})^B{}_a \tag{2.8}$$

(See Marsden and Hughes [1983], p. 31.)

Acceleration Vectors

In addition, associated with the motion φ_t, we define the ***material acceleration*** $\mathbf{A}_t$ and the ***spatial acceleration*** $\mathbf{a}_t$ by the expressions

$$\mathbf{A}_t = \partial^2\varphi_t/\partial t^2 = \partial \mathbf{V}_t/\partial t \quad \text{and} \quad \mathbf{a}_t = \mathbf{A}_t \circ \varphi^{-1} \tag{2.9}$$

The ***convected acceleration*** $\mathcal{A}_t$ is defined by the pull-back relation

$$\mathcal{A}_t = \varphi_t^*(\mathbf{a}_t) \tag{2.10}$$

Proposition 2.2. *The convected velocity and acceleration are related by the formula*

$$\mathcal{A}_t = \frac{\partial \mathcal{V}_t}{\partial t} + \nabla_{\mathcal{V}_t} \mathcal{V}_t \tag{2.11a}$$

In coordinates, $\mathcal{A}$ *is given by*

$$\mathcal{A}^A = \partial \mathcal{V}^A / \partial t + \mathcal{V}^C \mathcal{V}^A{}_{,C} + \Gamma_{CD}{}^A \mathcal{V}^C \mathcal{V}^D \tag{2.11b}$$

Proof. Recall (Marsden and Hughes [1983] p. 33) that the spatial velocity and accelerations are related by

$$a_t = \frac{\partial v_t}{\partial t} + \nabla_{v_t} v_t \quad . \tag{2.12}$$

Now pull back the relation (2.12) by φ_t; one gets

$$\varphi_t^*(\nabla_{v_t} v_t) = \nabla_{\varphi_t^* v_t} \varphi_t^* v_t = \nabla_{\mathcal{V}} \mathcal{V}_t,$$

and by the Lie derivative formula (Marsden and Hughes [1983], §1.6),

$$\frac{\partial \mathcal{V}_t}{\partial t} = \frac{\partial}{\partial t}(\varphi_t^* v_t) = \varphi_t^*(L_{v_t} v_t) + \varphi_t^*\left(\frac{\partial v_t}{\partial t}\right) = \varphi_t^*\left(\frac{\partial v_t}{\partial t}\right) \tag{2.13}$$

Adding (2.12) and (2.13) gives (2.11). ■

The Stress Tensor and Covariance

We assume the existence of a stored energy function $W : \mathcal{M}_{\mathcal{S}} \times \mathcal{C} \times \mathcal{M}_{\mathcal{B}} \longrightarrow \mathbb{R}$, where $\mathcal{M}_{\mathcal{S}}$ is the space of Riemannian metrics on $\mathcal{S}$, and $\mathcal{M}_{\mathcal{B}}$ is the space of Riemannian metrics on $\mathcal{B}$, of the form

$$W = \overline{W}(g, F, G) \tag{2.14}$$

where $\overline{W}$ depends only on the point values of g, F, and G. This is in keeping with the

classical assumption that the stored energy function in an elastic material depends on the configuration φ only locally through the point values of the deformation gradient F (see Marsden and Hughes [1983], §§3.2 and 3.3). The dependence of the stored energy $\overline{W}$ on the metric tensor g is essential to introduce the notion of covariance. Consider an arbitrary superposed spatial diffeomorphism $\eta : \mathcal{S} \longrightarrow \mathcal{S}$; then we say that $\overline{W}$ is ***covariant*** if

$$\overline{W}(g, F, G) \circ \eta = \overline{W}(\eta_* g, T\eta \circ F, G) \tag{2.14}$$

This assumption implies that $\overline{W}$ depends only on the point values of the Cauchy-Green tensor C. That is, there is a function of the point values of metrics on $\mathcal{B}$, which we shall denote by $\overline{\overline{W}}$ such that

$$\overline{W}(g, F, G) = \overline{\overline{W}}(C, G) \tag{2.15}$$

Let σ be the ***Cauchy stress tensor*** and let $\tau = \mathrm{Jac}(\varphi)\sigma$, (where $\mathrm{Jac}(\varphi)$ is the Jacobian determinant of φ), be the ***Kirchhoff stress tensor***. Finally, the ***symmetric Piola-Kirchhoff stress tensor*** is defined by the pull-back relation

$$S = \varphi^*(\tau) = \mathrm{Jac}(\varphi)\varphi^*(\sigma) \tag{2.16}$$

We then have the constitutive relations

$$\tau = 2\rho_{ref}\,\frac{\partial \overline{W}(g, F, G)}{\partial g} \qquad \text{and} \qquad S = 2\rho_{ref}\,\frac{\partial \overline{\overline{W}}(C, G)}{\partial C}\,, \tag{2.17}$$

where ρ_{ref} is the density in the reference configuration. The relation (2.17_1) is referred to as the spatial ***Doyle-Ericksen formula*** (see Marsden and Hughes [1983] §3.3, and, for the material counterpart, Simo and Marsden [1984]). In terms of the ***Lagrangian strain tensor*** $E = (C - G)/2$, and the ***Eulerian strain tensor*** $e = \varphi_*(E) = (g - b^{-1})/2$, where $b^{-1} = \varphi_*(G)$ is the ***Finger deformation tensor***, formulae (2.17) read

$$\tau = \rho_{ref}\,\frac{\partial \overline{W}(e, F, G)}{\partial e} \qquad \text{and} \qquad S = \rho_{ref}\,\frac{\partial \overline{\overline{W}}(E, G)}{\partial E} \tag{2.18}$$

Note that the dependence of W on the material metric tenso G has been explicitly assumed in equation (2.14) but that G is treated as a parameter as far as the covariance assumption is concerned.

Next, we consider the invariance group of W on the left and on the right. We observe that the form of the stored energy function $W = \overline{W}(g, F, G)$ is left invariant under the action of the group $\mathrm{Diff}(\mathcal{S})$ of all diffeomorphisms of $\mathcal{S}$ onto $\mathcal{S}$. Indeed, this follows directly from the covariance assumption. Thus, we regard the function $\overline{W}(C, G)$ as being obtained by *left reduction* from the function $\overline{W}$. This relation is key to the derivation of the Hamiltonian structure in the convective representation and will be explored in detail in the next section.

The Convective Hamiltonian

We conclude this section with a discussion of the Hamiltonian in the convected picture. In the Hamiltonian formalism for the material picture, the kinetic and potential energy are expressed in the variables on $T^*\mathcal{C}$; i.e., in the variables $\varphi \in \mathcal{C}$, and its conjugate momentum $M_\varphi \in T^*\mathcal{C}$, which is related to the material velocity by the coordinate formula

$$M_a(X) = g_{ab}(x)V^b(X)\rho_{ref}(X)\, d^3X \tag{2.19}$$

so that M_φ is a one form density covering φ. We obtain the expressions for the kinetic energy in the convective representation by simply changing variables as follows. The square length of the material velocity is given by

$$\begin{aligned} \|V_t(X)\|^2 &= V^a(X)V^b(X)g_{ab}(x) \qquad (\text{where } x = \varphi_t(X)) \\ &= F^a{}_A F^b{}_B \mathcal{V}^A \mathcal{V}^B g_{ab}(x) \\ &= C_{AB}\mathcal{V}^A\mathcal{V}^B. \end{aligned} \tag{2.20}$$

Introducing the inner product

$$\langle \mathcal{V}, \mathcal{W} \rangle := \int_{\mathcal{B}} \rho_{ref}\, C_{AB}\mathcal{V}^A\mathcal{V}^B d^3X \tag{2.21}$$

on the space of convective velocity fields, the kinetic energy may be written as

$$\begin{aligned} K &= \frac{1}{2}\int_{\mathcal{B}} \rho_{ref} V^a V^b g_{ab}\, d^3X \\ &= \frac{1}{2}\langle \mathcal{V}, \mathcal{V} \rangle \end{aligned} \tag{2.22}$$

This kinetic energy induces a function on the cotangent bundle in a standard manner. The corresponding momentum induced by this kinetic energy via the Legendre transform is given by

$$\mathcal{M} := \rho_{ref} \mathcal{V}^{\flat} d^3X \tag{2.23}$$

where $\flat$ denotes the index lowering operation so that $\mathcal{V}^{\flat}$ is given in coordinates by $\mathcal{V}_A = C_{AB}\mathcal{V}^B$, and d^3X is the coordinate volume element on $\mathcal{B}$. Similarly, following standard notation (see, for example, Abraham, Marsden, and Ratiu [1983]), the index raising operation is denoted by $\#$, so that $\mathcal{M}^{\#}$ is given in coordinates by $\mathcal{M}^A = C^{AB}\mathcal{M}_B$, where C^{AB} denotes the inverse matrix to C_{AB}. Substitution of (2.23) into (2.22) gives

$$K = \frac{1}{2} \int_{\mathcal{B}} \frac{1}{\rho_{ref}} \mathcal{M}_A \mathcal{M}_B C^{AB} d^3X \tag{2.24}$$

Note that K is now a function of $\mathcal{M}$ and C alone. We also note that the kinetic energy is just one half the square length of the momentum density in the metric on the space of convective momentum densities that is induced by (2.21). Thus the total Hamiltonian is given by the expression

$$H(\mathcal{M}, C, G) = \frac{1}{2} \int_{\mathcal{B}} \frac{1}{\rho_{ref}} \mathcal{M}_A \mathcal{M}_B C^{AB} d^3X + \int_{\mathcal{B}} \rho_{ref} \overline{W}(C, G)\, d^3X \tag{2.25}$$

We think of this form of the energy as being the energy induced on the original space of φ, M, g's after quotienting by the group of spatial diffeomorphisms $\mathrm{Diff}(\mathcal{S})$ (and again G enters parametrically). This idea is central to the reduction procedure that will be explained in the next section.

Convected Equations of Motion

The convected equations are obtained by either pulling back the equations of motion in spatial coordinates or by a coordinate calculation of $\partial\mathcal{M}/\partial t$, etc. From balance of linear momentum, one gets the following:

$$\dot{\mathcal{M}} = C \cdot \mathrm{Div}_C S + [\nabla \mathcal{V}]^T \mathcal{M} \tag{2.26a}$$

whereas balance of angular momentum yields the standard symmetry condition $S = S^T$. Finally

one gets the additional relation:

$$\dot{C} = L_{\mathcal{V}} C \tag{2.26b}$$

Here, $L_{\mathcal{V}} C$ denotes the Lie derivative along the flow of the convected velocity field $\mathcal{V}_t$. Equation (2.26b) follows directly from the relation between flows and Lie derivatives and the relation $C_t = \varphi_t{}^*(g)$. Another argument is as follows. Recalling (2.4c) and (2.6a), we have

$$L_{\mathcal{V}} C = L_{\varphi_t^*(v)} \varphi^*(C) = \varphi_t^*(L_v g) \tag{2.27}$$

Since $L_v g = 2d$, where d is the rate of deformation tensor, (2.26b) follows from (2.27) by recalling the well known relation

$$\dot{C} = 2\varphi_t^*(d).$$

It is equations (2.26a) and (2.26b) that we shall show are Hamiltonian in the next section.

§3. *Hamiltonian Structure of Three Dimensional Elasticity in the Convective Representation*

In this section we show that the equations of elastodynamics in the convective representation are Hamiltonian relative to a non-canonical Poisson structure on the space of pairs $(\mathcal{M}, \mathbf{C})$, where $\mathcal{M}$ is the convective momentum density and $\mathbf{C}$ is the Cauchy-Green tensor, as in the preceeding section. This means that the equations of elastodynamics are equivalent to this condition: for any function $f(\mathcal{M}, \mathbf{C})$,

$$\dot{f} = \{f, H\}, \tag{3.1}$$

where H is the Hamiltonian, given by equation (2.25). The bracket $\{\ \}$ satisfies the usual conditions for a Poisson bracket, including Jacobi's identity (see, for example, Marsden et al. [1983] for some generalities).

The bracket $\{\,,\}$ appearing in (3.1) will be obtained by reducing the canonical bracket on $T^*\mathcal{C}$ by the group of spatial isometries of the metric $\mathbf{g}$ on $\mathcal{S}$. Equivalently, as in Marsden, Ratiu, and Weinstein [1984a,b], we can add the metric $\mathbf{g}$ as a parameter and reduce $T^*(\mathcal{C} \times \mathcal{M}_{\mathcal{S}})$ by the left action of $\mathrm{Diff}(\mathcal{S})$. This reduction procedure will be explicitly explained by direct calculation in what follows. Before reading this section, the reader may find it helpful to first review the parallel case of rigid body dynamics in §4.

The Canonical Bracket on the Material Phase Space

We start with the canonical bracket on $T^*\mathcal{C}$, the space of deformations φ and their canonically conjugate momenta M_φ, the material momentum densities. Now let f and g be functions of the pairs $(\mathcal{M}, \mathbf{C})$. Define the new functions $\overline{f}, \overline{g}$ on $T^*\mathcal{C}$ by

$$\overline{f}(\varphi, M_\varphi) = f(\varphi^* M_\varphi, \varphi^* \mathbf{g}) \tag{3.2}$$

where

$$\varphi^* M_\varphi =: \mathcal{M} = (T\varphi)^T \cdot M_\varphi \tag{3.3}$$

Here, $(T\varphi)^T = F^T$ is the transpose of the deformation gradient and

$$\varphi^* g = C = F^T F \tag{3.4}$$

as before. In coordinates, (3.3) reads

$$\mathcal{M}_A(X) = F^a{}_A(X) M_a(X) \tag{3.3'}$$

and

$$C_{AB}(X) = F^a{}_A(X) F^b{}_B(X) g_{ab}(x), \tag{3.4'}$$

where $x = \varphi(X)$. We define $\overline{g}$ in a similar way. Now we form the canonical bracket on $T^*\mathcal{C}$

$$\{\overline{f}, \overline{g}\} = \int_{\mathcal{B}} \left(\frac{\delta \overline{f}}{\delta \varphi} \cdot \frac{\delta \overline{g}}{\delta M_\varphi} - \frac{\delta \overline{g}}{\delta \varphi} \cdot \frac{\delta \overline{f}}{\delta M_\varphi} \right) d^3X \tag{3.5}$$

where $\delta \overline{f} / \delta \varphi$ is the functional derivative. (This Poisson bracket is written as if φ and M_φ were independent variables; this requires a little caution since $T^*\mathcal{C}$ is not simply a product space. However, it is well known how to deal with this situation; see Abraham and Marsden [1978] and Marsden and Hughes [1983]). The bracket (3.5) is now to be expanded using the chain rule and the result expressed in terms of f, g, $\mathcal{M}$, and C. This will produce the desired reduced bracket. General reduction theory (Marsden et al. [1983], Marsden and Ratiu [1986] and references therein) shows that the elastodynamic equations, which are Hamiltonian on $T^*\mathcal{C}$ (or Lagrangian on $T\mathcal{C}$), reduce to Hamiltonian equations (3.1) on our convective phase space $\mathcal{P}$, the space of $(\mathcal{M}, C)$'s.

Before proceeding with this calculation, we make some additional remarks. First, if $\mathcal{C}$ were a group, we would expect the reduced bracket to be of a special type, namely a Lie-Poisson bracket on the dual of the Lie algebra of a semi-direct product; see Marsden, Ratiu, and Weinstein [1984a,b]. We shall see analogues of such a structure here. Because the boundary of $\mathcal{B}$ can move, $\mathcal{C}$ is not a group and indeed our bracket differs from a semi-direct product Lie-Poisson bracket only in boundary integral terms. For another bracket in *spatial* representation, see Lewis, Marsden, Montgomery, and Ratiu [1986]; see also Holm, Marsden, and Ratiu [1986].) The boundary conditions here can either be displacement, traction, or a combination. In the former case we restrict $\mathcal{C}$, which imposes corresponding restrictions on $\mathcal{V}$ and $\mathcal{M}$; specifically, if a portion of $\partial\mathcal{B}$ is fixed, $\mathcal{M}$ will vanish on that portion. For traction boundary conditions we add a corresponding boundary term to the Hamiltonian (as in Marsden and Hughes [1983], for example). These boundary terms are to be taken into account in the calculation of functional derivatives, as in Lewis, Marsden, Montgomery, and Ratiu [1986]).

A second remark is this. The bracket in *purely* spatial terms seems to be possible only for

isotropic elasticity; see Marsden, Ratiu, and Weinstein [1984a,b]. No such restriction is needed for the convective representation. The reason for this discrepancy is simply that material frame indifference gives us a covariance symmetry with which to reduce by $\mathrm{Diff}(\mathcal{S})$. The corresponding reduction by $\mathrm{Diff}(\mathcal{B})$ requires isotropy of the material.

A third remark is that given a dynamic solution $\mathcal{M}(X, t), C(X, t)$, the original motion $\varphi(X, t)$ can be reconstructed by constructing $\mathcal{V}$ using (2.23) and then solving the ordinary differential equation for Proposition 2.1:

$$\frac{\partial \psi(x, t)}{\partial t} = \mathcal{V}(\psi(x, t), t) \tag{3.6a}$$

and letting

$$\varphi_t = \psi_t^{-1}. \tag{3.6b}$$

This is a special case of the general reconstruction procedure that is used in reduction theory. (See Abraham and Marsden [1978]).

The reduced Poisson structure is given in the following theorem. The notation is as follows:

(i) $\delta f/\delta\mathcal{M}$ is the vector field on $\mathcal{B}$ such that for all variations $\delta\mathcal{M}$ (which are also a one form density, like $\mathcal{M}$),

$$Df\cdot\delta\mathcal{M} := \frac{d}{d\varepsilon} f(\mathcal{M} + \varepsilon\delta\mathcal{M}, C)\Big|_{\varepsilon=0} = \int_{\mathcal{B}} \frac{\delta f}{\delta\mathcal{M}}\cdot\delta\mathcal{M}\, d^3X. \tag{3.7a}$$

(ii) $\delta f/\delta C$ is the symmetric two tensor (indices up) density on $\mathcal{B}$ such that

$$Df\cdot\delta C := \frac{d}{d\varepsilon} f(\mathcal{M}, C + \varepsilon\delta C)\Big|_{\varepsilon=0} = \int_{\mathcal{B}} \frac{\delta f}{\delta C} : \delta C\, d^3X \tag{3.7b}$$

for all variations δC (which is a symmetric two tensor, with indices down, on $\mathcal{B}$.)

(iii) $\mathrm{DIV}(\cdot)$ denotes the divergence relative to the coordinate chart $\{X^A\}$ with associated (reference configuration) metric $G_{AB}(X)$, whereas $\mathrm{Div}_C(\cdot)$ denotes the divergence operator relative to the convected metric $C_{AB}(X)$. In addition, covariant differentiation relative to the convected metric C_{AB} is denoted in the remainder of this section simply by ∇ or in coordinates with a vertical bar; for instance, $\mathrm{Div}_C\mathcal{V} = \mathcal{V}^A{}_{|A} = \mathrm{tr}[\nabla\mathcal{V}]$.

Let us denote by $[\cdot]^A$ the skew-symmetric part of a second rank tensor $[\cdot]$. Then we define the ***convected vorticity*** $\mathcal{W}$ by

$$\frac{1}{2}\mathcal{W} \times H = [\nabla \mathcal{M}]^A H \text{ , for all vectors } H \in \mathbb{R}^3 \text{ .} \tag{3.8}$$

Theorem 3.1 *The reduced Poisson bracket on* $\mathcal{P}$ *is given by*

$$\{f, g\} = -\int_{\mathcal{B}} \Big\{ \mathcal{W}\cdot\Big[\frac{\partial f}{\partial \mathcal{M}} \times \frac{\partial g}{\partial \mathcal{M}}\Big]$$

$$+ \mathcal{M}\cdot\Big[\mathrm{Div}_C\Big(\frac{\partial f}{\partial \mathcal{M}}\Big)\frac{\partial g}{\partial \mathcal{M}} - \mathrm{Div}_C\Big(\frac{\partial g}{\partial \mathcal{M}}\Big)\frac{\partial f}{\partial \mathcal{M}}\Big]$$

$$+ C : \Big[\mathrm{Div}_C\Big(2\frac{\partial f}{\partial C}\Big) \otimes \frac{\partial g}{\partial \mathcal{M}} - \mathrm{Div}_C\Big(2\frac{\partial g}{\partial C}\Big) \otimes \frac{\partial f}{\partial \mathcal{M}}\Big]\Big\} d^3x \tag{3.9a}$$

$$= \int_{\mathcal{B}} \mathcal{M} \cdot \Big[\frac{\delta f}{\delta \mathcal{M}}, \frac{\delta g}{\delta \mathcal{M}}\Big] d^3X$$

$$- \int_{\partial\mathcal{B}} \mathcal{M}\cdot\Big[\frac{\delta g}{\delta \mathcal{M}}\Big(\frac{\delta f}{\delta \mathcal{M}}\cdot N\Big) - \frac{\delta f}{\delta \mathcal{M}}\Big(\frac{\delta g}{\delta \mathcal{M}}\cdot N\Big)\Big] dA$$

$$+ \int_{\mathcal{B}} C : \Big[\mathcal{L}_{\frac{\delta f}{\delta \mathcal{M}}}\Big(\frac{\delta g}{\delta C}\Big) - \mathcal{L}_{\frac{\delta g}{\delta \mathcal{M}}}\Big(\frac{\delta f}{\delta C}\Big)\Big] d^3X$$

$$+ \int_{\partial\mathcal{B}} C : \Big[\frac{\delta g}{\delta C}\Big(\frac{\delta f}{\delta \mathcal{M}}\cdot N\Big) - \frac{\delta f}{\delta C}\Big(\frac{\delta g}{\delta \mathcal{M}}\cdot N\Big)\Big] dA \tag{3.9b}$$

where N *is the outward unit normal on* $\partial\mathcal{B}$, $\mathcal{L}$ *is the Lie derivative (recall that* $\delta g/\delta C$ *is a two tensor density with indices up), and where* [,] *denotes the usual Lie bracket of vector fields.*

Note that for displacement boundary conditions, $(\delta f/\delta \mathcal{M})\cdot N = 0$, and so the two boundary terms in (3.9b) disappear, resulting in a Lie- Poisson bracket for the semi-direct product

$\mathfrak{X}(\mathcal{B}) \circledS$ (2-tensor densities).

We turn now to the proof of the theorem. For simplicity of notation, we just write f for $\overline{f}$.

Lemma 3.2. *The following formulae hold:*

$$\frac{\delta f}{\delta \varphi^a} = -g_{ab}F^b{}_A\left[2\frac{\partial f}{\partial C_{AB}} + \mathcal{M}^A \frac{\partial f}{\partial \mathcal{M}_B}\right]_{|B} \tag{3.10a}$$

$$\frac{\delta f}{\delta M_a} = F^a{}_A \frac{\partial f}{\partial \mathcal{M}_A} . \tag{3.10b}$$

Proof. The second formula follows at once from the chain rule. To prove the first one we proceed as follows. Denoting by $\langle \cdot , \cdot \rangle$ the standard L_2-pairing, we have

$$\left\langle \frac{\delta f}{\delta \varphi^a}, \delta\varphi^a \right\rangle = \int_{\mathcal{B}} \left[2\frac{\partial f}{\partial C_{AB}} F^a{}_A \delta\varphi^b{}_{,B}\, g_{ab} + \frac{\partial f}{\partial \mathcal{M}_A} \delta\varphi^a{}_{,A}\, M_a\right] dV$$

$$= \int_{\mathcal{B}} g_{ab}F^b{}_A\left[2\frac{\partial f}{\partial C_{AB}} + \mathcal{M}^A \frac{\partial f}{\partial \mathcal{M}_B}\right]\delta\varphi^a{}_{,B}\; dV \tag{3.11}$$

where use has been made of the relation

$$M_a = (F^{-1})^C{}_a\mathcal{M}_C = (F^{-1})^C{}_a C_{CA}\mathcal{M}^A = g_{ab}F^b{}_A\mathcal{M}^A . \tag{3.12}$$

Next, recall that if P and S are related through $P^{aA} = F^a{}_B S^{BA}$, then one has

$$\mathrm{DIV}\, P = T\varphi \cdot \mathrm{Div}_C S . \tag{3.13}$$

Using the divergence theorem (and assuming $\delta\varphi^a$ is zero on $\partial\mathcal{B}$) we have

$$\left\langle \frac{\partial f}{\partial \varphi^a}, \delta\varphi^a \right\rangle = \int_{\mathcal{B}} -g_{ab}F^b{}_A\left[2\frac{\partial f}{\partial C_{AB}} + \mathcal{M}^A \frac{\partial f}{\partial \mathcal{M}_B}\right]_{|B}\delta\varphi^a\, dV \tag{3.14}$$

Hence

$$\frac{\delta f}{\delta \varphi} = -g^{\flat} \cdot F \cdot \mathrm{Div}_C \Big[2 \frac{\partial f}{\partial C^{\flat}} + \mathcal{M}^{\#} \otimes \frac{\partial f}{\partial \mathcal{M}} \Big] . \tag{3.15}$$

which completes the proof. ■

Making use of this proposition, and the identity $C_{AB|B} = 0$, we have

$$\Big\langle \frac{\delta f}{\delta \varphi^a}, \frac{\delta g}{\delta M_a} \Big\rangle = \int_{\mathcal{B}} - \frac{\partial g}{\partial \mathcal{M}_A} \Big\{ C_{AB} \Big[2 \frac{\partial f}{\partial C_{BC}} \Big]_{|C} + \Big[\mathcal{M}_A \frac{\partial f}{\partial \mathcal{M}_C} \Big]_{|C} \Big\} \, dV \tag{3.16}$$

Thus

$$\begin{aligned} \{f, g\} = \int_{\mathcal{B}} \Big\{ &- \Big[\frac{\partial g}{\partial \mathcal{M}} \otimes \mathrm{Div}_C\Big(2 \frac{\partial f}{\partial C}\Big) - \frac{\partial f}{\partial \mathcal{M}} \otimes \mathrm{Div}_C\Big(2 \frac{\partial g}{\partial C}\Big) \Big] : C \\ &- \Big[\mathrm{Div}_C\Big(\frac{\partial f}{\partial \mathcal{M}}\Big) \frac{\partial g}{\partial \mathcal{M}} - \mathrm{Div}_C\Big(\frac{\partial g}{\partial \mathcal{M}}\Big) \frac{\partial f}{\partial \mathcal{M}} \Big] \cdot \mathcal{M}^{\flat} \\ &- \Big[\frac{\partial g}{\partial \mathcal{M}} \otimes \frac{\partial f}{\partial \mathcal{M}} - \frac{\partial f}{\partial \mathcal{M}} \otimes \frac{\partial g}{\partial \mathcal{M}} \Big] : \nabla \mathcal{M} \Big\} \, d^3X . \end{aligned} \tag{3.17}$$

Since $\hat{\theta} : A = \hat{\theta} : [A]^A$ for any $A \in GL(\mathbb{R}^3)$ (see equation (3.8)), and $\hat{\theta} \in so(3)$, (the skew symmetric 3×3 matrices), we have

$$\begin{aligned} \nabla \mathcal{M} : \Big[\frac{\partial g}{\partial \mathcal{M}} \otimes \frac{\partial f}{\partial \mathcal{M}} - \frac{\partial f}{\partial \mathcal{M}} \otimes \frac{\partial g}{\partial \mathcal{M}} \Big] &= [\nabla \mathcal{M}]^A : \Big[\frac{\partial f}{\partial \mathcal{M}} \times \frac{\partial g}{\partial \mathcal{M}} \Big]^{\wedge} \\ &= w \cdot \Big[\frac{\partial f}{\partial \mathcal{M}} \times \frac{\partial g}{\partial \mathcal{M}} \Big] \end{aligned} \tag{3.18}$$

where $^\wedge : \mathbb{R}^3 \longrightarrow so(3)$ is the usual isomorphism (recalled explicitly in section 4).

Substituting (3.18) into (3.17) gives (3.9a). Formula (3.9b) is obtained from (3.9a) by integration by parts (the divergence theorem). ■

Corollary 3.3. *Hamilton's equations* $\dot{f} = \{f, H\}$ *are equivalent to the convected equations*

$$\rho_{\mathrm{ref}}(\dot{\mathcal{V}} + \nabla_{\mathcal{V}}\mathcal{V}) = \mathrm{Div}_C \mathbf{S} \tag{3.19a}$$

[or

$$\dot{\mathcal{M}} = C\cdot\mathrm{Div}_C \mathbf{S} + [\nabla\mathcal{V}]^T\mathcal{M} \tag{3.19a'}$$]

and

$$\dot{C} = \mathcal{L}_{\mathcal{V}} C \tag{3.19b}$$

This follows, as was remarked earlier, by the general theory of reduction. It also can be checked by a direct calculation.

§4. *The Rotation Group and Rigid Body Dynamics*

In what follows we summarize some basic notation and elementary properties of the rotation group needed for subsequent developments. For a more detailed account we refer the reader to standard textbooks, such as Abraham and Marsden [1982, §4.1] and Choquet and DeWitt-Morette [1982, pp. 181-194]. We then review the Hamiltonian structure of rigid body dynamics.

The Rotation Group and its Lie Algebra

Following standard usage, we denote by $SO(3)$ the Lie group of proper orthogonal transformations, i.e.,

$$SO(3) := \left\{ \Lambda : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \;\middle|\; \Lambda \text{ is linear}, \; \Lambda^T\Lambda = 1, \text{ and } \det \Lambda = 1 \right\}; \tag{4.1}$$

$SO(3)$ is a compact subgroup of the general linear group $GL(3)$. Its Lie algebra is $so(3)$ $:= \{\hat{\Theta} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \mid \hat{\Theta} \text{ is linear and } \hat{\Theta} + \hat{\Theta}^T = 0\}$, the set of all skew-symmetric tensors. The notation is as follows: In coordinates, relative to an orthonormal basis $\{e_i\}$ in $\mathbb{R}^3$, we write $\hat{\Theta} = \hat{\Theta}^i{}_j e_i \otimes e^j$, $\Theta = \Theta^i e_i$, and, in matrix notation,

$$[\hat{\Theta}^i{}_j] = \begin{bmatrix} 0 & -\Theta^3 & \Theta^2 \\ \Theta^3 & 0 & -\Theta^1 \\ -\Theta^2 & \Theta^1 & 0 \end{bmatrix}, \quad \{\Theta^i\} = \begin{bmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{bmatrix} \tag{4.2}$$

Recall that $so(3)$ and $\mathbb{R}^3$ are related through the isomorphism $\hat{\ } : \mathbb{R}^3 \longrightarrow so(3)$, defined by the relation $\hat{\Theta}h = \Theta \times h$, for any $h \in \mathbb{R}^3$. Here, $\Theta \in \mathbb{R}^3$ is the axial vector of $\hat{\Theta} \in so(3)$, and $\times$ denotes the ordinary vector product. Physically, $\Lambda \in SO(3)$ represents a finite rotation while infinitesimal rotations are linearized rotations about the identity. Geometrically, one speaks of $so(3)$ as the tangent space of $SO(3)$ at the identity $1 \in SO(3)$, and employs the notation $so(3) = T_1SO(3)$.

Left and Right Representations of the Tangent Space to SO(3)

Given any $\Lambda \in SO(3)$, elements $\hat{\Theta}_\Lambda$ of the tangent space $T_\Lambda SO(3)$ at a point

$\Lambda \in SO(3)$ are represented in two alternative forms:

(i) *Left invariant vector fields* defined by the relation $\hat{\Theta}_\Lambda := T_I L_\Lambda \hat{\Theta}$, where $\hat{\Theta} \in so(3)$, and $L_\Lambda : SO(3) \longrightarrow SO(3)$ denotes left translation; i.e., $L_\Lambda \Lambda_1 := \Lambda\Lambda_1$. A simple calculation yields $\hat{\Theta}_\Lambda = \Lambda\hat{\Theta}$. Accordingly, we have the following identification of $T_\Lambda SO(3)$:

$$T_\Lambda SO(3) = \{\hat{\Theta}_\Lambda := \Lambda\hat{\Theta} \mid \text{for any } \hat{\Theta} \in so(3)\} \ . \tag{4.3a}$$

Geometrically, an element $\hat{\Theta}_\Lambda \in T_\Lambda SO(3)$ corresponds to a finite rotation superimposed onto an infinitesimal rotation $\hat{\Theta} \in so(3)$.

(ii) *Right invariant vector fields.* The characterization is identical to that in (i), with left translations replaced by right translations $R_\Lambda : SO(3) \longrightarrow SO(3)$ defined as $R_\Lambda\Lambda_1 = \Lambda_1\Lambda$. This leads to the representation

$$T_\Lambda SO(3) := \{\hat{\theta}_\Lambda := \hat{\theta}\Lambda \mid \text{for any } \hat{\theta} \in so(3)\} \ . \tag{4.3b}$$

Geometrically, an element $\hat{\theta}_\Lambda \in T_\Lambda SO(3)$ represents an infinitesimal rotation superimposed onto a finite rotation represented by Λ. Following the general conventions we use in elasticity, left invariant vector fields at a point Λ are denoted by upper case letters; i.e., $\hat{\Theta}_\Lambda$. We think of $\hat{\Theta} \in so(3)$, with axial vector $\Theta \in R^3$, as a *material object* with coordinates Θ^I relative to a material basis $\{E_I\}$. In the context of rigid body dynamics one often speaks of *body representations* and *body coordinates* relative to the body frame $\{t_I\}$ defined by $t_I := \Lambda E_I$. On the other hand, right invariant vector fields at a point Λ are denoted by lower case letters; i.e., $\hat{\theta}_\Lambda$. One thinks of $\hat{\theta} \in so(3)$, with axial vector $\theta \in R^3$, as a spatial object and coordinates θ^i relative to a spatially fixed basis $\{e_i\}$. In the context of rigid body mechanics one speaks of the *spatial representation*.

The Dual of the Lie Algebra

We also recall that $so(3)^*$, the cotangent space at the identity $I \in SO(3)$, is isomorphic to $[R^3, \times]$ and to $so(3)$ via the dot product; that is, we also identify elements of $so(3)^*$ with skew symmetric matrices and use the map $\hat{}$ and the pairing given by

$$\hat{\Pi}(\hat{W}) = \frac{1}{2}\,\hat{\Pi}:W \equiv \Pi\cdot W\ , \tag{4.4}$$

where $\hat{\Pi}:\hat{W} = \mathrm{trace}\,[\Pi^T W]$, and we have used the notation $A:B = A^I{}_J\, B_I{}^J$ for any A,B $\in GL(3)$. Thus, the duality pairing

$$\langle\cdot\,,\cdot\rangle : T^*_{\Lambda}SO(3) \times T_{\Lambda}SO(3) \longrightarrow \mathbb{R} \tag{4.5a}$$

is defined by

$$\langle\hat{\Pi}_{\Lambda},\, \hat{W}_{\Lambda}\rangle := \frac{1}{2}\,\mathrm{trace}\,[\Pi^T_{\Lambda}\hat{W}_{\Lambda}] \equiv \frac{1}{2}\,\hat{\Pi}_{\Lambda}:W_{\Lambda} \tag{4.5b}$$

Note that this duality pairing is *left invariant* in the sense that

$$\langle\hat{\Pi}_{\Lambda},\, \hat{W}_{\Lambda}\rangle := \frac{1}{2}\,\mathrm{trace}\left[\hat{\Pi}^T\Lambda^T\Lambda\hat{W}\right] \equiv \frac{1}{2}\,\hat{\Pi} : \hat{W} = \langle\hat{\Pi},\hat{W}\rangle \tag{4.5c}$$

The Exponential Map

Finally, we recall that the straight line $\varepsilon \mapsto \varepsilon\hat{\theta} \in T_I SO(3)$, for $\varepsilon > 0$, is mapped by the *exponential map* onto the curve

$$\varepsilon \mapsto \exp[\varepsilon\hat{\theta}] = \Big[\sum_{k\geq 0} \frac{\varepsilon^k}{k!}\,\hat{\theta}^k\Big] \in SO(3)\ . \tag{4.6}$$

Note that for the case of $\hat{\theta} \in so(3)$ one has the following explicit formula often credited to Rodrigues (Goldstein [1980, p. 165]):

$$\exp\,[\hat{\theta}] = I + \frac{\sin|\theta|}{|\theta|}\,\hat{\theta} + \frac{1}{2}\,\frac{\sin^2(|\theta|/2)}{(|\theta|/2)^2}\,\hat{\theta}^2 \tag{4.7}$$

This formula is of fundamental importance in the numerical solution of initial boundary value problems for finite strain rods (Simo and Vu-Quoc [1986a]).

Let us now recall the Hamiltonian structure of rigid body dynamics in body (= convective)

representation. This is done purely for the reader's convenience to see the parallel with the developments in the preceding and subsequent sections. It is also useful to keep this parallel in mind for the problem of coupled dynamics (Krishnaprasad, Marsden, and Simo [1986]). For further details on rigid body dynamics in this context, see Marsden, Ratiu, and Weinstein [1984b].

The configuration space is $\mathcal{C} = SO(3)$, and the Hamiltonian $H : T^*SO(3) \longrightarrow \mathbb{R}$ is given by

$$H = \frac{1}{2} \Pi \cdot \mathbb{I}^{-1} \Pi . \tag{4.8}$$

Here, $\mathbb{I}$ is the *inertia dyadic* defined in terms of the Lagrangian coordinates $X = (X^1, X^2, X^3)$ as

$$\mathbb{I} := \int_\Omega \rho_{ref}(X) \left[\|X\|^2 I - X \otimes X \right] d^3X , \tag{4.9}$$

and Π is the ***body momentum*** defined as

$$\Pi := \mathbb{I} \left[\Lambda^T \quad \dot{\Lambda} \right]^{V} . \tag{4.10}$$

We observe that H is *invariant under spatial isometries*; that is, invariant under the left action of $SO(3)$. *Reduction* by this symmetry amounts to considering functions $f : T^*SO(3) \longrightarrow \mathbb{R}$ of the form

$$f(\Lambda, \hat{\Pi}_\Lambda) = \bar{f}(\Lambda^T \hat{\Pi}_\Lambda) \equiv \bar{f}(\hat{\Pi}) . \tag{4.11}$$

The canonical bracket on $T^*SO(3)$ is given in terms of the duality pairing $\langle \cdot , \cdot \rangle :$ $T^*_\Lambda SO(3) \times T_\Lambda SO(3) \longrightarrow \mathbb{R}$ defined by (4.5) as

$$\{f, g\} = \frac{1}{2} \left[\left\langle \frac{\partial f}{\partial \Lambda} , \frac{\partial g}{\partial \hat{\Pi}_\Lambda} \right\rangle - \left\langle \frac{\partial g}{\partial \Lambda} , \frac{\partial f}{\partial \hat{\Pi}_\Lambda} \right\rangle \right] \tag{4.12}$$

The *Lie-Poisson* bracket on the reduced space $\mathcal{P} = T^*SO(3)/SO(3)$ is obtained with the aid of the results given below

Proposition 4.1. *The following formulae hold*

$$\frac{\partial \bar{f}}{\partial \Lambda} = -\frac{1}{2}\,\Lambda\left[\Pi \times \frac{\partial \tilde{f}}{\partial \Pi}\right]^{\wedge} . \tag{4.13}$$

$$\frac{\partial \bar{f}}{\partial \hat{\Pi}_{\Lambda}} = \Lambda\left[\frac{\partial \tilde{f}}{\partial \Pi}\right]^{\wedge} . \tag{4.14}$$

where $\bar{f}(\hat{\Pi}) = \tilde{f}(\Pi)$.

Proof. To prove (4.13), observe that for any $\delta\Lambda \in T_{\Lambda}SO(3)$ we have the left representation $\delta\Lambda = \Lambda\delta\hat{\Theta}$, where $\delta\hat{\Theta} \in so(3)$. Thus, by the chain rule and the left invariance of the duality pairing we have

$$\left\langle \frac{\partial f}{\partial \Lambda}, \delta\Lambda \right\rangle = \left\langle \frac{\partial \bar{f}}{\partial \hat{\Pi}}, \delta\Lambda^{T}\,\hat{\Pi}_{\Lambda} \right\rangle$$

$$= \left\langle -\frac{\partial \bar{f}}{\partial \hat{\Pi}}, \delta\hat{\Theta}\,\hat{\Pi} \right\rangle$$

$$= \left\langle -\hat{\Pi}\,\frac{\partial f}{\partial \hat{\Pi}}, \delta\hat{\Theta} \right\rangle$$

Since $\dfrac{\partial \bar{f}}{\partial \hat{\Pi}} \in so(3)$, it follows that

$$\left\langle \frac{\partial f}{\partial \Lambda}, \delta\Lambda \right\rangle = \left\langle -\frac{1}{2}\left[\hat{\Pi}, \frac{\partial \bar{f}}{\partial \hat{\Pi}}\right], \delta\hat{\Theta} \right\rangle$$

So that, on setting $\bar{f}(\hat{\Pi}) = \tilde{f}(\Pi)$ and recalling the Lie bracket relation

$$\hat{A}\hat{B} - \hat{B}\hat{A} = [A \times B]^{\wedge},$$

we obtain

$$\left\langle \frac{\partial f}{\partial \Lambda}, \delta\Lambda \right\rangle = \left\langle -\frac{1}{2}\,\Lambda\left[\Pi \times \frac{\partial \tilde{f}}{\partial \Pi}\right]^{\wedge}, \delta\Lambda \right\rangle,$$

and result (4.13) follows. An analogous calculation holds for the formula (4.14).■

Substitution of (4.13) and (4.14) into (4.12), and use of standard vector product identities yields the result

$$\{f,\, g\} \;=\; -\,\Pi\cdot\left[\,\frac{\partial\tilde{f}}{\partial\Pi} \times \frac{\partial\tilde{g}}{\partial\Pi}\,\right]\,. \tag{4.16}$$

which is the standard Lie-Poisson bracket for rigid body dynamics. The equation $\dot{f} = \{f,\, H\}$ is then easily checked to be equivalent to Euler's equation for rigid body dynamics: $\dot{\Pi} = \Omega \times \Pi$, where $\hat{\Omega} = \Lambda^T\dot{\Lambda}$ is the body angular velocity.

§5. *Geometrically Exact Finite Strain Rod*

The static version of the rod model summarized below goes back essentially to Reissner [1973] who modified the classical Kirchhoff - Love model (see Love [1944]) to account for shear deformation. An equivalent model, formulated as a constrained director theory - the so-called special theory of Cosserat rods - is due to Antman [1974] - see also Antman and Jordan [1975], Antman and Kenny [1981], and Antman [1984] for some applications. The dynamic version along with the parametrization discussed below is given in Simo [1985]. For completeness, a brief account is outlined next.

The Configuration Space

From a physical standpoint, the configurations of a rod deforming in the ambient space $\mathbb{R}^3$ may be defined by specifying : (i) The position of its line of centroids by means of the map $\varphi : [0, L] \longrightarrow \mathbb{R}^3$, and (ii) The orientation of cross sections at points $S \in [0, L]$. This can be done using the orientation of a moving basis $\{t_I(S) \mid I = 1,2,3\}$ attatched to the cross section relative to a fixed frame $\{E_I \mid I = 1,2,3\}$, referred to as a *material frame* in what follows. The moving basis is described by means of an orthogonal transformation $\Lambda : [0, L] \longrightarrow SO(3)$ such that $t_I(S) = \Lambda(S)E_I$.

If we view the rod as having a finite cross section given by a compact set $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial\Omega$, then the placement of the rod in its reference configuration is determined by a map $\Phi_0(S) = (\varphi_0(S), \Lambda_0(S))$ in such a way that the corresponding set ocupied by the rod is given by

$$\mathcal{B} = \left\{ X \in \mathbb{R}^3 \,\middle|\, X = \varphi_0(S) + \sum_{\alpha=1}^{2} \xi^\alpha \Lambda_0(S) E_\alpha, \text{ where } (\xi^1, \xi^2, S) \in \Omega \times [0, L] \right\} \tag{5.1a}$$

Without loss of generality we can assume that $\|\varphi_0'(S)\| = 1$, so that L is the length of the reference line of centroids. Typically, one chooses $\Lambda_0(S)$ so that $T_I(S) := \Lambda_0(S)E_I$ is the Frenet frame associated with the curve φ_0; that is,

$$T_3 := \varphi_0', \quad T_1 := \varphi_0''/\|\varphi_0''\| \quad \text{and} \quad T_2 := T_3 \times T_1 \tag{5.1b}$$

According to these ideas, we take for our configuration space the set

$$\mathcal{C} = \{\Phi = (\varphi, \Lambda) \mid [0, L] \rightarrow \mathbb{R}^3 \times SO(3)\} \tag{5.2}$$

For simplicity, we shall assume that φ and Λ satisfy pure displacement boundary conditions in what follows; i.e., $\varphi|_{S=0,L}$ and $\Lambda|_{S=0,L}$ are prescribed. Hence, the tangent space at the identity configuration is given by

$$\begin{aligned} T_{\text{identity}}\mathcal{C} \\ = \{(\delta\varphi, \delta\hat{\Theta}) : [0, L] \rightarrow \mathbb{R}^3 \times so(3) \mid \delta\varphi|_{S=0,L} = 0 \\ \text{and } \delta\hat{\Theta}|_{S=0,L} = 0\}. \end{aligned} \tag{5.3}$$

Left and right invariant tangent vector fields at a configuration $\Phi \in \mathcal{C}$ are defined below in the standard fashion by employing left and right translations. Finally, associated to any configuration $\Phi \in \mathcal{C}$ one defines its *arc-length* by the mapping

$$S \in [0, L] \mapsto s = \tilde{s}(S) := \int_0^S \left\| \frac{\partial}{\partial \xi}\varphi(\xi) \right\| d\xi \tag{5.4}$$

The arc length may be used to parametrize points on the center line of the current configuration. For the convective description, this is not necessary, but it is convenient in the spatial description. We shall tacitly assume that the rod does not self-intersect; that is, that on the image of the above mapping, there is a well defined smooth inverse mapping $x \mapsto S$, where x is a point in the image of Φ. The image curve is parametrized by the arc length and so we will regard the role of x as being played by s in what follows.

Motions and Velocity Fields

A ***motion*** is a curve of configurations $t \in [0,T] \mapsto \Phi_t = (\varphi_t, \Lambda_t) \in \mathcal{C}$ for some time interval $[0,T]$. Associated with a motion, one has the ***material velocity field*** $V_\Phi(S,t)$ defined by

$$V_\Phi(S,t) := \frac{\partial}{\partial t}\Phi(S,t) := (\dot{\varphi}_t(S, t), \dot{\Lambda}_t(S, t)) \tag{5.4}$$

Thus at any time t, the material velocity is an element of $T_\Phi\mathcal{C}$, the tangent space to $\mathcal{C}$ at the

configuration Φ at time t. The *spatial velocity field* $V_\Phi(s,t)$ is defined by

$$V_\Phi(s,t) := (\dot{\phi}(S,t), \dot{\Lambda}(S,t)), \tag{5.5b}$$

where, as above, $s = \tilde{s}(S,t)$ is the arc length in the current configuration. Finally, the ***convected velocity field*** $\mathcal{V}_\Phi(S,t)$ is defined by the expression

$$\mathcal{V}_\Phi(S,t) := ([\Lambda(S,t)]^T\dot{\phi}(S,t), [\Lambda(S,t)]^T\dot{\Lambda}(S,t)) \tag{5.5c}$$

Since $\Lambda(S,t) \in SO(3)$, we can write

$$\dot{\Lambda}(S,t) = \Lambda(S,t)\,\hat{W}(S,t) = \hat{w}(s,t)\,\Lambda(S,t) \tag{5.6}$$

where $W(S,t) \in so(3)$ and $w(s,t) \in so(3)$. Accordingly, the convected and spatial velocity fields can be expressed as

$$\mathcal{V}_\Phi(S,t) = ([\Lambda(S,t)]^T\,\dot{\phi}(S,t), \hat{W}(S,t))$$

and

$$V_\Phi(s,t) = (\dot{\phi}(S,t), \hat{w}(s,t)) \tag{5.7}$$

Remark. One can easily check that the components of the spatial velocity in the moving frame $\{t_I(S)\}$ are equal to the components of the convected velocity in the inertial frame $\{E_I\}$. We also remark that the material velocity field V_Φ may be viewed either as a left extension of the convected velocity field $\mathcal{V}_\Phi$ or as a right extension of the spatial velocity field V_Φ.

Strain Measures.

As in the three dimensional theory, one defines the (one-dimensional) ***deformation gradient*** as

$$\Phi'(S,t) := (\varphi'(S,t), \Lambda'(S,t)), \quad \text{where} \quad (\cdot)' := \partial/\partial S \tag{5.8}$$

Similarly, in parallel with the definition of velocity fields, starting from the (Lagrangian) deformation gradient $\Phi'(S,t)$ one defines ***convected and spatial strains*** according to

convected	*material*	*spatial*
$\Gamma(S,t) := [\Lambda(S,t)]^T\cdot\varphi'(S,t)$	$\varphi'(S,t)$	$\gamma(s,t) := \partial\varphi(S,t)/\partial s$
$\Omega(S,t) := [\Lambda(S,t)]^T\cdot\Lambda'(S,t)$	$\Lambda'(S,t)$	$\omega(s,t) := [\partial\Lambda(S,t)/\partial s]\cdot\Lambda^T(S,t)$

where $s = \tilde{s}(S,\ t)$. We again note that the components of the convected strains in the material frame $\{E_I\}$ coincide with the components of the spatial strains in the moving frame $\{t_I(S)\}$ up to a factor (because of the arc length parametrization that may be used in the spatial description.) The above expressions can be derived from the three dimensional theory by a duality argument employing the formula for the stress power given in the remark below.

The Equations of Motion in Spatial Description

Associated with the motion $t \mapsto \Phi_t \in C$, one assumes the existence of smooth vector fields $n(s,t)$, $m(s,t)$, and a scalar field $\rho(s,t)$ interpreted respectively as the contact resultant force, contact resultant couple, and density per unit of current length. These fields satisfy the following *spatial local form* of the equations of motion:

$$\dot{\rho} + (\dot{J}J^{-1})\frac{\partial\rho}{\partial s} = 0$$

$$\rho A v = \frac{\partial n}{\partial s} + \bar{n} \tag{5.10}$$

$$\rho[\mathbb{J}\dot{w} + w \times \mathbb{J}w] = \frac{\partial m}{\partial s} + \gamma \times n + \bar{m}$$

In these equations, we use the following notation:

$$\dot{} = \frac{\partial}{\partial t} + \frac{\partial}{\partial s}\frac{\partial\tilde{s}}{\partial t}$$

denotes the material time derivative, and $\bar{n}$, m are the body force and torque per unit of current length $s = \tilde{s}(S,\ t)$. The role of the Jacobian is played by $J = \partial\tilde{s}/\partial S$; $\mathbb{J}(s,t)$ is the time

dependent inertia dyadic of the cross section given, relative to the moving frame $\{\mathbf{t}_I\}$ by

$$\mathbb{J} := \mathrm{J}^{\alpha\beta}\, \mathbf{t}_\alpha \otimes \mathbf{t}_\beta + \mathrm{J}_{33}\mathbf{t}_3 \otimes \mathbf{t}_3 \tag{5.11}$$

where $\mathrm{J}^{\alpha\beta}$ and J_{33} in (5.11) and A in (5.10) are inertia constants associated with the reference cross section. (Typically $\mathrm{J}^{\alpha\beta} := \int_\Omega \xi_\alpha \xi_\beta \, d\Omega$, $\alpha, \beta = 1,2$, and $\mathrm{A} := \int_\Omega d\Omega$, where ξ_1, ξ_2 are coordinates in the reference configuration defined by (5.1).)

Equations (5.10) are the local statement in the spatial description of balance of mass, balance of linear momentum and balance of angular momentum, respectively. We refer to Antman [1972],[1976] for a derivation of analogous equations from the three dimensional theory.

Equations of Motion in the Convective Description

In the convective description, the balance laws are expressed directly in the inertial frame $\{\mathbf{E}_I\}$. To this end one defines vector fields $\mathbf{N}(S,t)$, $\mathbf{M}(S,t)$, and $\rho_{ref}(S)$ as pull-backs of their spatia counterparts by means of the relations

$$\begin{aligned} \mathbf{N}(S,t) &:= [\Lambda(S,t)]^T \cdot \mathbf{n}(s,t), \\ \mathbf{M}(S,t) &:= [\Lambda(S,t)]^T \cdot \mathbf{m}(s,t), \\ \rho_{ref}(S) &:= J(S,t)\rho(s,t) \end{aligned} \tag{5.12}$$

where $s = \tilde{s}(S,t)$ and $J(S,t) = \partial\tilde{s}(S,t)/\partial S$. Omitting explicit indication of the arguments, by making use of the relation

$$\frac{\partial \mathbf{m}}{\partial s} = J^{-1}\, \Lambda[\mathbf{M}' + \boldsymbol{\Omega} \times \mathbf{M}] \tag{5.13}$$

and the analogous expression connecting $\mathbf{n}$ and $\mathbf{N}$, a straight forward calculation then yields the following statements of balance of mass, linear and angular momentum:

$$\begin{aligned} \frac{\partial \rho_{ref}}{\partial t} &= 0 \\ A\rho_{ref}[\dot{\mathcal{V}} + \mathbf{W} \times \mathcal{V}] &= \frac{\partial \mathbf{N}}{\partial S} + \boldsymbol{\Omega} \times \mathbf{N} + \bar{\mathbf{N}} \end{aligned} \tag{5.14}$$

$$\rho_{ref}[\mathbf{J}\dot{\mathbf{W}} + \mathbf{W} \times \mathbf{J}\mathbf{W}] = \frac{\partial \mathbf{M}}{\partial S} + \boldsymbol{\Omega} \times \mathbf{M} + \boldsymbol{\Gamma} \times \mathbf{N} + \overline{\mathbf{M}}$$

where $\mathbf{V} = \Lambda^T\dot{\varphi}$ is the convective velocity and $\mathbf{J} := \Lambda^T \mathbf{j} \Lambda$ is the *time independent* inertia dyadic.

Remark: Within the context of the three dimensional theory, $\mathbf{n}$, and $\mathbf{m}$ are defined as the resultant force and the resultant torque relative to the line of centroids of the distribution of stress acting on a cross section. The definition of $\boldsymbol{\gamma}$ and $\boldsymbol{\omega}$ is unique in the sense that the stress power is given by

$$\int_{\Omega\times[0,L]} \mathbf{P}:\dot{\mathbf{F}}\, d\Gamma\, dS = \int_{\varphi([0,L])} [\mathbf{n}\cdot\boldsymbol{\gamma}^{\nabla} + \mathbf{m}\cdot\boldsymbol{\omega}^{\nabla}]\, ds = \int_{[0,L]} \mathbf{N}\cdot\dot{\boldsymbol{\Gamma}} + \mathbf{M}\cdot\dot{\boldsymbol{\Omega}}]\, dS$$

(5.15a)

Here, $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, $\mathbf{F}$ is the deformation gradient of the configuration given by

$$\Phi = \varphi(S,t) + \sum_{\alpha=1}^{2} \xi^{\alpha} \mathbf{t}_{\alpha}(S,t), \qquad \text{and} \qquad {}^{\nabla} = {}^{\cdot} - \mathbf{w} \times \tag{5.15b}$$

is the co-rotated rate measuring the rate of change relative to the moving frame.

Appropriate stress measures conjugate to the Lagrangian strains (φ', Λ') may also be obtained by either (i) left extension of the spatial stress measure $(\mathbf{n},\mathbf{m})$ or (ii) right extension of the convected stress measures $(\mathbf{N},\mathbf{M})$. For instance, the latter extension takes the form

$$\hat{\mathbf{N}}_{\Lambda} := \Lambda\hat{\mathbf{N}} \quad \text{and} \quad \hat{\mathbf{M}}_{\Lambda} := \Lambda\hat{\mathbf{M}} \tag{5.16}$$

Obviously, such an extension preserves the stress power. ■

Constitutive Equations

In view of the expression (5.15) for the stress power, one characterizes (isothermal) hyperelastic response by assuming the existence of a free energy function $\psi(S,\boldsymbol{\gamma},\boldsymbol{\omega},\Lambda)$ such that

$$\mathbf{n} = \frac{\partial\psi(S,\boldsymbol{\gamma},\boldsymbol{\omega},\Lambda)}{\partial\boldsymbol{\gamma}},$$

(5.17)

and $$m = \frac{\partial\psi(S,\gamma,\omega,\Lambda)}{\partial\omega}.$$

By postulating that these equations are frame indifferent; i.e.,covariant under the left action of the Euclidean group of spatial isometries, a further reduction is obtained as follows. Let $t \mapsto Q(t) \in SO(3)$ be an arbitrary superposed rigid body rotation, and let $\Phi_t^* := (Q(t)\varphi, Q(t)\Lambda_t) \in C$. Denoting by $(\cdot)^*$ objects associated with Φ_t^*, one has the relations

$$\begin{aligned} \gamma^* &= Q(t)\gamma, \quad n^* = Q(t)n \\ \omega^* &= Q(t)\omega, \quad m^* = Q(t)m \\ \psi^* &= \psi \end{aligned} \tag{5.18}$$

Thus, by choosing $Q = \Lambda^T$, and since $\psi^*(S, Q\gamma, Q\omega, Q\Lambda) = \psi(S, \gamma, \omega, \Lambda)$, it follows that the only possible form compatible with frame indifference and locality (see, for example, Marsden and Hughes [1983]) is given by

$$N = \frac{\partial\Psi(S,\Gamma,\Omega)}{\partial\Gamma} \quad \text{and} \quad M = \frac{\partial\Psi(S,\Gamma,\Omega)}{\partial\Omega} \tag{5.19}$$

Example. An example of a constitutive equation consistent with the above invariance requirements and useful in computation (see for example, Simo and Vu-Quoc [1986b]) is furnished by the ***uncoupled linear systems :***

$$N = C_N(\Gamma - \Gamma^0) \; ; \qquad M = C_M(\Omega - \Omega^0) \tag{5.20}$$

where C_N and C_M are symmetric positive definite and constant, $\Gamma^0 = \Lambda_0^T\varphi_0'$ and $\Omega^0 = \Lambda_0^T\Lambda_0'$. This ensures that the reference configuration $\Phi_0(S) := (\varphi_0(S), \Lambda_0(S))$ is stress free. Linear constitutive models of the type (5.20) are analogous to the Saint Venant-Kirchhoff model of three dimensional elasticity and are typically restricted to small strains due to the behavior of ψ. We refer to Ciarlet [1986, Chapter 4] for a summary of appropriate growth conditions of ψ.

Hamiltonian in the convective description

One defines convected linear and angular momenta according to the following expressions

$$\mathcal{M} := \rho_{ref} A \mathcal{V} \qquad \text{and} \qquad \Pi := \rho_{ref} J \mathcal{W} \tag{5.21}$$

Thus, in the absence of body forces and couples, and assuming for simplicity pure displacement boundary conditions, the Hamiltonian is given by

$$H((\Gamma, \Omega); (\mathcal{M}, \Pi)) =$$

$$\int_0^L ((A\ \rho_{ref})^{-1} \|\mathcal{M}\|^2 + \Pi \cdot (\ \rho_{ref} J)^{-1} \Pi) dS$$

$$+ \int_0^L \psi(S, \Gamma, \Omega) dS \tag{5.22}$$

In the next section we shall consider the Hamiltonian structure of this rod model on the reduced space $\mathcal{P} = T^* \mathcal{C} / SO(3)$.

§6. *The Hamiltonian Structure for the Geometrically Exact Rod Model in the Convective Picture*

In this section we shall derive the Poisson structure which makes the equations for a geometrically exact rod model Hamiltonian. To do this we shall proceed in a way that is similar to that in three dimensional elasticity and the free rigid body, namely, we shall reduce the canonical bracket in material representation by means of spatial isometries. To carry this out, begin by recalling that the configuration space for the rod model is given by

$$\mathcal{C} = \{ \Phi \mid \Phi = (\varphi, \Lambda) : [0, L] \longrightarrow \mathbb{R}^3 \times SO(3) \} \tag{6.1}$$

The tangent space to $\mathcal{C}$ at a configuration Φ is given by

$$T_\Phi \mathcal{C} = \{ V_\Phi \mid V_\Phi = (V_\varphi, \hat{W}_\Lambda) : [0, L] \longrightarrow \mathbb{R}^3 \times T_\Lambda SO(3) \} \tag{6.2}$$

and the cotangent space is given by

$$T^*_\Phi \mathcal{C} = \{ M_\Phi = (M_\varphi, \hat{\Pi}_\Lambda) : [0, L] \longrightarrow \mathbb{R}^3 \times T^*_\Lambda SO(3) \} \tag{6.3}$$

Now introduce the duality pairing

$$\langle \cdot, \cdot \rangle : T_\Phi \mathcal{C} \times T^*_\Phi \mathcal{C} \longrightarrow \mathbb{R}$$

defined as follows:

$$\langle V_\Phi, M_\Phi \rangle = \int_0^L [M_\varphi \cdot V_\varphi + \tfrac{1}{2} \hat{\Pi}_\Lambda : \hat{W}_\Lambda] \, dS \tag{6.4}$$

Recall that the reason for the factor 1/2 is that according to (4.4) and (4.5),

$$\hat{\Pi}_\Lambda : \hat{W}_\Lambda = 2\, \Pi \cdot W \tag{6.5}$$

Next, we turn our attention to the formulation of the canonical Hamiltonian structure on $T^*\mathcal{C}$.

The Canonical Bracket

We consider functions $f, g : T_\Phi^* C \longrightarrow \mathbb{R}$ so that the canonical Poisson bracket is given by

$$\{f, g\} := \left\langle \frac{\delta f}{\delta \Phi} \cdot \frac{\delta g}{\delta M_\Phi} \right\rangle - \left\langle \frac{\delta g}{\delta \Phi} \cdot \frac{\delta f}{\delta M_\Phi} \right\rangle . \tag{6.6}$$

Explicitly, one has the expression

$$\{f, g\} = \int_0^L \left[\frac{\delta f}{\delta \varphi} \cdot \frac{\delta g}{\delta M_\varphi} - \frac{\delta g}{\delta \varphi} \cdot \frac{\delta f}{\delta M_\varphi}\right] dS$$

$$+ \frac{1}{2} \int_0^L \left[\frac{\delta f}{\delta \Lambda} : \frac{\delta g}{\delta \Pi_\Lambda} - \frac{\delta g}{\delta \Lambda} : \frac{\delta f}{\delta \Pi_\Lambda}\right] dS . \tag{6.7}$$

As in section 3, we remark that the Poisson bracket (6.7) is written as if the variables (φ, M_φ) and (Λ, Π_Λ) were both independent. This however, requires some caution since T^*C is not simply a produxt space. Again, we refer to Abraham and Marsden [1978].

Reduced (Poisson) Bracket

In the present context, reduction by *material frame indifference* amounts to considering functions $f : T^*C \longrightarrow \mathbb{R}$ which are $SO(3)$ invariant; these are of the following form:

$$f(\Phi; M_\Phi) \equiv f((\varphi, \Lambda); (M_\varphi, \Pi_\Lambda))$$

$$= \bar{f}((\Lambda^T\varphi', \Lambda^T\Lambda'); (\Lambda^T M_\varphi, \Lambda^T \Pi_\Lambda)) , \tag{6.8}$$

where $(\cdot)' \equiv d(\cdot)/dS$. Introducing the notation

$$\left.\begin{aligned} \Gamma &:= \Lambda^T \varphi' \\ \hat{\Omega} &:= \Lambda^T \Lambda' \end{aligned}\right|$$

$$\left.\begin{aligned} \mathcal{M} &:= \Lambda^T M_\varphi \\ \Pi &:= \Lambda^T \Pi_\Lambda \end{aligned}\right\} \tag{6.9}$$

which is consistent with the left representation of $T_\Phi C$ we may rewrite equation (6.8) as

$$\begin{aligned} f((\varphi, \Lambda); (M_\varphi, \Pi_\Lambda)) &= \bar{f}((\Gamma, \hat{\Omega}); (\mathcal{M}, \Pi)) \\ &= \tilde{f}((\Gamma, \Omega); (\mathcal{M}, \Pi)) \ . \end{aligned} \tag{6.10}$$

One now obtains a *reduced* bracket in terms of the variables $((\Gamma, \Omega); (\mathcal{M}, \Pi))$ by transforming the canonical bracket with the aid of the chain rule. The key result is contained in the following:

Proposition 6.1. *We have the following formulae:*

$$\frac{\delta f}{\delta \Pi_\Lambda} = \Lambda\left[\frac{\partial \tilde{f}}{\partial \Pi}\right]^\wedge \ ; \tag{6.11a}$$

$$\frac{\delta f}{\delta \varphi} = -\Lambda\left[\left(\frac{\partial \tilde{f}}{\partial \Gamma}\right)' + \Omega \times \frac{\partial \tilde{f}}{\partial \Gamma}\right] \ ; \tag{6.11b}$$

$$\frac{\delta \tilde{f}}{\delta M_\varphi} = \Lambda \frac{\partial \tilde{f}}{\delta \mathcal{M}} \ ; \tag{6.11c}$$

$$\frac{\delta f}{\delta \Lambda} = -\Lambda\left[\Gamma \times \frac{\partial \tilde{f}}{\partial \Gamma} + \mathcal{M} \times \frac{\partial \tilde{f}}{\partial \mathcal{M}} + \frac{1}{2}\Pi \times \frac{\partial \tilde{f}}{\partial \Pi} + \left(\frac{\partial \tilde{f}}{\partial \Omega}\right)' + \Omega \times \frac{\partial \tilde{f}}{\partial \Omega}\right] . \tag{6.11d}$$

Proof. The first three formulae above follow directly from the chain rule. To prove the last one, we proceed as follows. Making use of the chain rule we have

$$\left\langle \frac{\delta f}{\delta \Lambda}, \delta\Lambda \right\rangle = \int_0^L \left\{ \frac{\partial \bar{f}}{\partial \Gamma} \cdot \delta\Lambda^T \varphi' + \frac{1}{2} \operatorname{tr}\left[\frac{\partial \bar{f}}{\partial \hat{\Omega}} (\delta\Lambda^T \Lambda' + \Lambda^T \delta\Lambda')\right]\right.$$

$$+ \frac{\partial \bar{f}}{\partial \mathcal{M}} \cdot \delta\Lambda^T \varphi' + \frac{1}{2} \operatorname{tr}\left[\frac{\partial \bar{f}}{\partial \Pi} \delta\Lambda^T \Pi_\Lambda \right\} dS \tag{6.12}$$

$$= \int_0^L \Lambda^T \delta\Lambda : \left[\Gamma \otimes \frac{\partial \bar{f}}{\partial \Gamma} + \mathcal{M} \otimes \frac{\partial \bar{f}}{\partial \mathcal{M}} - \frac{1}{2} \hat{\Omega} \frac{\partial \bar{f}}{\partial \hat{\Omega}} \right] dS$$

$$- \int_0^L \frac{1}{2} \delta\Lambda : \left[\left(\Lambda \frac{\partial \bar{f}}{\partial \hat{\Omega}} \right)' + \Lambda \hat{\pi} \frac{\partial \bar{f}}{\partial \hat{\pi}} \right] dS , \tag{6.13}$$

where we have used skew-symmetry of $\partial \bar{f}/\partial \hat{\Omega}$, integration by parts, and the relation $\Lambda' = \Lambda \, \Omega$. We now observe that since $\delta\Lambda \in T_\Lambda SO(3)$ one has the left representation $\delta\Lambda = \Lambda \delta\hat{\Theta}$, where $\delta\hat{\Theta} \in so(3)$ and so we may rewrite (6.13) as

$$\left\langle \frac{\delta f}{\delta \Lambda} , \delta\Lambda \right\rangle = \int_0^L \delta\hat{\Theta} : \left[\Gamma \otimes \frac{\partial \bar{f}}{\partial \Gamma} + \mathcal{M} \otimes \frac{\partial \bar{f}}{\partial \mathcal{M}} - \hat{\Omega} \frac{\partial \bar{f}}{\partial \hat{\Omega}} \right.$$

$$\left. - \frac{1}{2} \Pi \frac{\partial f}{\partial \Pi} - \frac{1}{2} \left(\frac{\partial \bar{f}}{\partial \hat{\Omega}} \right)' \right] dS . \tag{6.14}$$

However, since $\delta\hat{\Theta} \in so(3)$, we have

$$\delta\hat{\Theta} : A \equiv \delta\hat{\Theta} : \left[\frac{A - A^T}{2} \right] \tag{6.15}$$

for any $A \in GL(\mathbb{R}^3)$. In addition, we recall the identity

$$[a \otimes b - b \otimes a] = - [a \times b]^\wedge , \quad a, b \in \mathbb{R}^3 , \tag{6.16}$$

along with the relation

$$[\hat{A}\hat{B} - \hat{B}\hat{A}] = [A \times B]^\wedge , \quad \hat{A}, \hat{B} \in so(3) . \tag{6.17}$$

Therefore, since $\partial \bar{f}/\partial \hat{\Omega} = [\partial \bar{f}/\partial \Omega]^\wedge$, we have

$$\left\langle \frac{\delta f}{\delta \Lambda}, \delta\Lambda \right\rangle = -\int_0^L \delta\hat{\Theta} : \frac{1}{2}\Big[\Gamma \times \frac{\partial \tilde{f}}{\partial \Gamma} + \mathcal{M} \times \frac{\partial \tilde{f}}{\partial \mathcal{M}} + \Omega \times \frac{\partial \tilde{f}}{\partial \Omega}$$

$$+ \frac{1}{2}\Pi \times \frac{\partial \tilde{f}}{\partial \Pi} + \Big(\frac{\partial \tilde{f}}{\partial \Omega}\Big)'\Big]^{\wedge} dS \; , \tag{6.18}$$

and replacing $\delta\hat{\Theta}$ by $\Lambda^T\delta\Lambda$, the result follows. ■

Using the formulae in Proposition 6.1 and employing the notation $\{f, g\} = \{f, g\}_{\varphi} + \{f, g\}_{\Lambda}$ where

$$\{f, g\}_{\Lambda} := \frac{1}{2}\int_0^L \Big(\frac{\partial f}{\partial \Lambda} : \frac{\partial g}{\partial \Pi_{\Lambda}} - \frac{\partial g}{\partial \Lambda} : \frac{\partial f}{\partial \Pi_{\Lambda}}\Big)\, dS \; , \tag{6.19a}$$

$$\{f, g\}_{\varphi} := \int_0^L \Big(\frac{\partial f}{\partial \varphi} \cdot \frac{\partial g}{\partial M_{\varphi}} - \frac{\partial f}{\partial M_{\varphi}} \cdot \frac{\partial g}{\partial \varphi}\Big)\, dS \; , \tag{6.19b}$$

we have

$$\{f, g\}_{\varphi} = -\int_0^L \Big[\Big(\frac{\partial \tilde{f}}{\partial \Gamma}\Big)' \cdot \frac{\partial \tilde{g}}{\partial \mathcal{M}} - \Big(\frac{\partial \tilde{g}}{\partial \Gamma}\Big)' \cdot \frac{\partial \tilde{f}}{\partial \mathcal{M}}\Big]\, dS$$

$$-\int_0^L \Omega \cdot \Big[\frac{\partial \tilde{f}}{\partial \Gamma} \times \frac{\partial \tilde{g}}{\partial \mathcal{M}} - \frac{\partial \tilde{g}}{\partial \Gamma} \times \frac{\partial \tilde{f}}{\partial \mathcal{M}}\Big]\, dS \; .$$

In addition, since $\hat{A} : \hat{B}/2 = A \cdot B$,

$\{f, g\}_{\Lambda} :=$

$$-\int_0^L \Big(\frac{1}{2}\Pi \times \frac{\partial \tilde{f}}{\partial \Pi} + \mathcal{M} \times \frac{\partial \tilde{f}}{\partial \mathcal{M}} + \Gamma \times \frac{\partial \tilde{f}}{\partial \Gamma} + \Omega \times \frac{\partial \tilde{f}}{\partial \Omega} + \Big(\frac{\partial \tilde{f}}{\partial \Omega}\Big)'\Big] \cdot \frac{\partial \tilde{g}}{\partial \Pi}\, dS$$

$$-\int_0^L \Big(\frac{1}{2}\Pi \times \frac{\partial \tilde{g}}{\partial \Pi} + \mathcal{M} \times \frac{\partial \tilde{g}}{\partial \mathcal{M}} + \Gamma \times \frac{\partial \tilde{g}}{\partial \Gamma} + \Omega \times \frac{\partial \tilde{g}}{\partial \Omega} + \Big(\frac{\partial \tilde{g}}{\partial \Omega}\Big)'\Big] \cdot \frac{\partial \tilde{f}}{\partial \Pi}\, dS \tag{6.21}$$

so that

$$\{f, g\}_\Lambda = -\int_0^L \Big[\Pi \cdot \Big(\frac{\partial \tilde f}{\partial \Pi} \times \frac{\partial \tilde g}{\partial \Pi}\Big) + \Big(\frac{\partial \tilde f}{\partial \Omega}\Big)' \cdot \frac{\partial \tilde g}{\partial \Pi} - \Big(\frac{\partial \tilde g}{\partial \Omega}\Big)' \cdot \frac{\partial \tilde f}{\partial \Pi}$$

$$+ \mathcal{M} \cdot \Big(\frac{\partial \tilde f}{\partial \mathcal{M}} \times \frac{\partial \tilde g}{\partial \Pi} - \frac{\partial \tilde g}{\partial \mathcal{M}} \times \frac{\partial \tilde f}{\partial \Pi}\Big) \tag{6.22}$$

$$+ \Omega \cdot \Big(\frac{\partial \tilde f}{\partial \Omega} \times \frac{\partial \tilde g}{\partial \Pi} - \frac{\partial \tilde g}{\partial \Omega} \times \frac{\partial \tilde f}{\partial \Pi}\Big)$$

$$+ \Gamma \cdot \Big(\frac{\partial \tilde f}{\partial \Gamma} \times \frac{\partial \tilde g}{\partial \Pi} - \frac{\partial \tilde g}{\partial \Gamma} \times \frac{\partial \tilde f}{\partial \Pi}\Big)\Big] dS$$

Thus, we have proved the following

Theorem 6.2. *The reduced Poisson bracket on* $\mathcal{P} = T^* C/SO(3)$ is given by

$$\{\tilde f, \tilde g\} = \begin{cases} -\int_0^L \Big\{\Big(\frac{\partial \tilde f}{\partial \Gamma}\Big)' \cdot \frac{\partial \tilde g}{\partial \mathcal{M}} - \Big(\frac{\partial \tilde g}{\partial \Gamma}\Big)' \cdot \frac{\partial \tilde f}{\partial \mathcal{M}} \\ \quad + \Big(\frac{\partial \tilde f}{\partial \Omega}\Big)' \cdot \frac{\delta \tilde g}{\partial \Pi} - \Big(\frac{\partial \tilde g}{\partial \Omega}\Big)' \cdot \frac{\partial \tilde f}{\partial \Pi}\Big\} dS & \textit{(canonical)} \\ -\int_0^L \Big\{\Omega \cdot \Big[\frac{\partial \tilde f}{\partial \Gamma} \times \frac{\partial \tilde g}{\partial \mathcal{M}} - \frac{\partial \tilde g}{\partial \Gamma} \times \frac{\partial \tilde f}{\partial \mathcal{M}}\Big]\Big\} dS & \textit{(interaction)} \\ -\int_0^L \Pi \cdot \Big[\frac{\partial \tilde f}{\partial \Pi} \times \frac{\partial \tilde g}{\partial \Pi}\Big] dS \\ + \int_0^L \Big\{\Omega \cdot \Big[\frac{\partial \tilde f}{\partial \Omega} \times \frac{\partial \tilde g}{\partial \Pi} - \frac{\partial \tilde g}{\partial \Omega} \times \frac{\partial \tilde f}{\partial \Pi}\Big] \\ \quad + \Gamma \cdot \Big[\frac{\partial \tilde f}{\partial \Gamma} \times \frac{\partial \tilde g}{\partial \Pi} - \frac{\partial \tilde g}{\partial \Gamma} \times \frac{\partial \tilde f}{\partial \Pi}\Big] & \textit{(Lie-Poisson for a semi-direct product)} \\ \quad + \mathcal{M} \cdot \Big[\frac{\partial \tilde f}{\partial \mathcal{M}} \times \frac{\partial \tilde g}{\partial \Pi} - \frac{\partial \tilde g}{\partial \mathcal{M}} \times \frac{\partial \tilde f}{\partial \Pi}\Big]\Big\} dS \end{cases} \tag{6.23}$$

Note that the *first integral* gives the *canonical term* in the variables $[(\Gamma, \Omega); (\mathcal{M}, \Pi)]$; the *second integral* gives *interaction terms*; and the *last two integrals* are the *Lie-Poisson terms*.

Corollary 6.3. *Hamilton's equations* $\dot{f} = \{ f, H \}$ *with Hamiltonian given by equation* (5.22) *and the Poisson bracket given by* (6.23) *are equivalent to the following convected equations of motion*

$$
\begin{aligned}
\dot{\mathcal{M}} &= N' + \Omega \times N - W \times \mathcal{M} \\
\dot{\Pi} &= M' + \Omega \times M + \Gamma \times N - W \times \Pi \\
\dot{\Gamma} &= \nu' + \Omega \times \nu - W \times \Gamma \\
\dot{\Omega} &= W' + \Omega \times W
\end{aligned}
\tag{6.24}
$$

where

$$N = \frac{\partial\psi}{\partial\Gamma} \quad \text{and} \quad M = \frac{\partial\psi}{\partial\Omega} \qquad \blacksquare$$

The last two equations in (6.24) can be checked by a direct calculation from the kinematic relations in section 5. The first two equations in (6.24) coincide with (5.14).

In summary, we have found the reduced space $\mathcal{P} = T^*\mathcal{C}/SO(3)$ to be the space of the convected variables $((\Gamma, \Omega); (\mathcal{M}, \pi))$ with the Poisson bracket given by (6.23). This reduced bracket has been obtained from the canonical bracket in material representation by reduction. As with three dimensional elasticity, if the motion on the reduced space is known, then the original motion in the material description can be reconstructed by solving the following system

$$\frac{\partial\Lambda}{\partial t} = \Lambda W \qquad \frac{\partial\varphi}{\partial t} = \Lambda\nu$$

Remark. The Hamiltonian formulation can be of assistance in the establishment of conservation laws, in addition to being useful for stability and bifurcation studies. For example, suppose that the rod has an isotropic cross section, so that the inertia dyadic J has two equal eigenvalues, just as in the case of the Lagrange top. Then there is a material symmetry group acting: it is the group S^1 acting on the *right*. Namely, let the symmetry group S^1 consist of rotations about the symmetry axis, say u. The group action is then given by the following action of $R \in S^1$: φ, and M_φ unchanged, and $(\Lambda, \Pi_\Lambda) \mapsto (\Lambda R, \Pi_\Lambda R)$. Here the momentum map corresponding to this action gives the conserved quantity

$$j = \int_0^L \Pi \cdot u \, dS.$$

§7. *Geometrically Exact Plate Model*

In this section we summarize the basic equations governing the goemetrically exact plate model. First, we introduce some necessary notation

Notation

In the geometric description of the configurations of a plate, the unit sphere

$$S^2 := \{t \in \mathbb{R}^3 \mid \|t\| = 1\} \quad . \tag{7.1a}$$

plays a central role. Given any $t \in S^2$, the tangent space at t is

$$T_tS^2 = \{v_t \in \mathbb{R}^3 \mid v_t \cdot t = 0\} \quad . \tag{7.1b}$$

Thus, T_tS^2 can be identified with $\mathbb{R}^2$. In addition to S^2 one introduces the following subset of $SO(3)$ which enables one to establish a link between the plate and rod formulations.

Let $E \in \mathbb{R}^3$ be a fixed but otherwise arbitrary vector. Define S_E to be the set of rotations $\Lambda \in SO(3)$ whose rotation axis $\psi \in \mathbb{R}^3$ is *normal* to E; i.e.,

$$S_E := \left\{\Lambda \in SO(3) \mid \psi \in \mathbb{R}^3 \text{ satisfies } \Lambda\psi = \psi \text{ and } \psi \cdot E = 0\right\}. \tag{7.2}$$

The set S_E is closely related to S, as the following (well-known) result shows.

Proposition 7.1. *Given any* $t \in S$ *there is one and only one element* $\Lambda_t \in S_E$ *such that*

$$t = \Lambda_t E \quad . \tag{7.3a}$$

In fact, S^2 *and* S_E *are diffeomorphic.*

Proof. Given $t \in S$ define the orthonormal set $\{E_I\}$ by

$$E_3 := E \ ,$$

$$E_1 := t \times E \ ,$$

$$E_2 := E_3 \times E_1 .$$

It follows that $E_2 = t - (E \cdot t)E$. Clearly, $\{E_I\}$ is uniquely defined for $t \neq E$. Let $\Theta := \cos^{-1}(E \cdot t)$ and define $\Lambda_t \in SO(3)$ by

$$\Lambda_t := E_1 \otimes E_1 + \cos\Theta(E_2 \otimes E_2 + E_3 \otimes E_3) + \sin\Theta(E_2 \otimes E_3 - E_3 \otimes E_2) .$$

Hence, $\Lambda_t \in S_E$. Uniqueness follows from the construction. ■

The geometric interpretation of this proposition should be clear. It constructs Λ_t by rotating E to t in the plane they span, through the angle Θ.

The tangent space to S_E at the identity is given by

$$T_I S_E := \{\hat{\theta} \in so(3) \mid \theta \cdot E = 0\} . \tag{7.3b}$$

Finally, the tangent space $T_\Lambda S_E$ at $\Lambda \in S_E$ is obtained by using either left or right tranlsations in $SO(3)$. For instance, for left invariant vector fields we have

$$T_\Lambda S_E = \{\hat{\theta}_\Lambda := \Lambda\hat{\theta} \mid \text{for any } \hat{\theta} \in T_I S_E, \text{ and } \Lambda \in S_E\}. \tag{7.3c}$$

An analogous characterization holds for right invariant fields.

Kinematic description of the plate

We consider the kinematic description of a plate with thickness $h > 0$. Essentially, no conceptual modification is required for the more general case of a shell. Let $\{E_I(X^0), I=1,2,3\}$ be a fixed *orthonormal* basis for $\mathbb{R}^3$, with $E_3 \equiv E$. Any point $X \in \mathbb{R}^3$ may be expressed as

$$X = X^0 + \xi E, \quad \text{where} \quad X^0 = X^\alpha E_\alpha, \ \xi \in \mathbb{R} . \tag{7.4a}$$

From the point of view of the three dimensional theory, the reference configuration of the plate is the set $\mathcal{B} \subset \mathbb{R}^3$ given by

$$\mathcal{B} \ := \ \left\{ X = X^0 + \xi E \ \middle| \ X^0 \in \Omega \ \text{ and } \ \xi \in \left[-\frac{h}{2}, \frac{h}{2} \right] \right\} , \qquad (7.4b)$$

where $\Omega \subset \mathbb{R}^2$ is a given region in the plane. One refers to $\{X^0 + \xi E \mid X^0 \in \partial\Omega$ and $\xi \in [-h/2, h/2]\}$ as the *edge* of the plate, and to Ω as its *mid-plane*. The basic kinematic assumption is that any *admissible configuration* $\tilde{\Phi} : \mathcal{B} \longrightarrow \mathbb{R}^3$ of the body is characterized as

$$x \ = \ \tilde{\Phi}(X) \ := \ \varphi(X^0) + \xi t(X^0) \quad , \qquad \xi \in \left[-\frac{h}{2}, \frac{h}{2} \right] . \qquad (7.5a)$$

where $\varphi : \Omega \longrightarrow \mathbb{R}^3$ maps the mid-plane onto $\varphi(\Omega) \in \mathbb{R}^3$, and t is a *unit vector* attached to the point $\varphi(X^0) \in \mathbb{R}^3$, not necessarily normal to $\varphi(\Omega)$, which is referred to as *director* or *fiber direction*. The fact that t need not be normal takes into account possible *shear* deformations.

Note that the mapping $X^0 \in \Omega \longrightarrow t(X^0) \in S^2$ assigns to points X^0, vectors t in the *unit sphere* (at $\varphi(X^0)$). By Proposition 7.1, the unit vector t at $\varphi(X^0)$ can be obtained from E *uniquely* through a rotation $\Lambda(X^0) \in S_E$ with rotation axis normal to E.

Configuration space

The kinematic assumption (7.3) embodies two essential ingredients: (i) Points $X^0 \in \Omega$ in the midplane are mapped onto points $x \in \mathbb{R}^3$ through the mapping $\varphi : \Omega \longrightarrow \mathbb{R}^3$; and (ii) *unit* vectors $E \in S^2$ attached to points $X^0 \in \Omega$ are mapped into *unit vectors* $t(X^0) \in S^2$ to points $\varphi(X^0) \in \mathbb{R}^3$ by rotations $\Lambda : \Omega \longrightarrow S_E$ with rotation axis normal to E. According to this view, two abstract characterizations of the set $\mathcal{C}$ of possible configurations of the plate are possible:

(a) *Director point of view:*

$$\mathcal{C} \ := \ \left\{ \overline{\Phi} \equiv (\varphi, t) \ \middle| \ \varphi : \Omega \longrightarrow \mathbb{R}^3 \ \text{ and } \ t : \Omega \longrightarrow S^2 \right\} . \qquad (7.6a)$$

The tangent space $T_{\overline{\Phi}}\,\mathcal{C}$ at a configuration $\overline{\Phi} \in \mathcal{C}$ is defined as

$$T_{\Phi}\,\mathcal{C} \ := \ \left\{ \delta\overline{\Phi} \equiv (\delta\varphi, \delta t) \ \middle| \ \delta\varphi : \Omega \longrightarrow \mathbb{R}^3 \ \text{ and } \ \delta t : \Omega \longrightarrow T_t S^2 \right\} . \qquad (7.6b)$$

(b) *Constrained frame point of view.* Equivalently, as a result of Proposition 7.1, the configuration space $\mathcal{C}$ may be defined as

$$\mathcal{C} := \left\{ \Phi \equiv (\varphi, \Lambda) \;\middle|\; \varphi : \Omega \rightarrow \mathbb{R}^3 \text{ and } \Lambda : \Omega \rightarrow S_E \right\} . \qquad (7.7a)$$

According to this view, the tangent space is

$$T_{\Phi}\mathcal{C} := \left\{ \delta\overline{\Phi} \equiv (\delta\varphi, \delta\Lambda) \;\middle|\; \delta\varphi : \Omega \rightarrow \mathbb{R}^3 \text{ and } \delta\Lambda : \Omega \rightarrow T_{\Lambda}S_E \right\}. \qquad (7.7b)$$

Points of view (a) and (b), although equivalent, lead to different forms of the equations of motion. Starting with the classical paper of Ericksen and Truesdell [1958], a substantial body of contemporary work, see, e.g., Antman [1976,1978], Green, Naghdi, and Wainwright [1965], and Naghdi [1972, 1980] has been typically concerned with the director point of view (a). Here, on the other hand, we take a different approach and adopt the constrained frame point of view (b) as a starting point. Our motivation for this lies in the strucure of the equations of motion and the Poisson bracket in approach (b) and which are essentially identical to that of the geometrically exact rod model considered in Sections 5 and 6. The form of the Poisson bracket corresponding to approach (b), in the convected description, is obtained by a reduction process that amounts to enforcing explicitly the additional (symmetry) condition that the rotation fields must lie in the subset $S_E \subset SO(3)$.

Motion and velocity fields

A ***motion*** is a curve of configurations $t \in [0, T] \rightarrow \Phi_t = (\varphi_t, \Lambda_t) \in \mathcal{C}$ for some time interval $[0, T] \subset \mathbb{R}_+$. Associated with the motion, one has the ***material velocity field*** $V_{\Phi}(X^0, t)$ defined as usual as

$$V_{\Phi}(X^0, t) := \frac{\partial}{\partial t}\Phi(X^0, t) \equiv (\dot{\varphi}_t(X^0), \dot{\Lambda}_t(X^0)) . \qquad (7.8a)$$

Thus the material velocity is a mapping $t \mapsto V_{\Phi_t} \in T_{\Phi}\,\mathcal{C}$, where $T_{\Phi}\mathcal{C}$ is the tangent space at configuration Φ. In addition we define the ***convected velocity*** field $\mathcal{V}_{\Phi}(X^0, t) \in \mathfrak{X}(\Omega)$ as in section 5, by the expression

$$\mathcal{V}_{\Phi}(X^0, t) := \left([\Lambda(X^0, t)]^T \dot{\varphi}_t(X^0, t), [\Lambda(X^0, t)]^T \dot{\Lambda}(X^0, t)\right) . \quad (7.8b)$$

Here, $\mathfrak{X}(\Omega)$ denotes the space of smooth fields $\mathcal{V}_{\Phi} : \Omega \longrightarrow \mathbb{R}^3 \times T_I S_E$. Since $\Lambda(X^0, t) \in S_E$, we have $\dot{\Lambda}(X^0, t) = \Lambda(X^0, t)\hat{W}(X^0, t)$ where $\hat{W}(X^0, t) \in T_I S_E$. Thus, the convected velocity field can be expressed as

$$\mathcal{V}_{\Phi}(X^0, t) = (\mathcal{V}, \hat{W})(X^0, t) \quad (7.8c)$$

where

$$\mathcal{V}(X^0, t) := [\Lambda(X^0, t)]^T V(X^0, t) , \quad V(X^0, t) := \dot{\phi}(X^0, t) .$$

Note that the material velocity field V_{Φ} is the *left extension* of the convected veclocity field $\mathcal{V}_{\Phi}$. Similarly, one could define a spatial field v_{Φ} whose second factor $w(X^0, t)$ is defined as

$$\hat{w}(X^0, t) := \dot{\Lambda}(X^0, t)[\Lambda(X^0, t)]^T . \quad (7.8d)$$

Then, $\hat{w} : \Omega \times [0, T] \longrightarrow T_I S_t$ is interpreted as a ***spatial angular velocity***, where $T_I S_t := \{\hat{\theta} \in so(3) \mid t \cdot \theta = 0\}$ is the tangent space at $t = \Lambda E$.

Strain measures

As in the geometrically exact rod model outlined in Section 5, starting from the Lagrangian deformation gradient $\Phi_{,\alpha}(X^0, t)$, $(\alpha = 1, 2)$, one defines *convected* strains according to the expressions

Convected	*Material*	*Spatial*
$\Gamma_\alpha(X^0,t) := \Lambda^T \varphi_{,\alpha}(X^0, t)$	$\varphi_{,\alpha}(X^0, t)$	$\gamma_\alpha = \varphi_{,\alpha}$
$\Omega_\alpha(X^0, t) = \left[\Lambda^T \Lambda_{,\alpha}(X^0, t)\right]^\vee$	$\Lambda_{,\alpha}(X^0, t)$	$\omega_\alpha = \left[\Lambda_{,\alpha}\Lambda^T\right]^\vee$

We note that the above expressions can also be derived from the three dimensional theory by a duality argument employing equivalence of the stress power. This is discussed briefly in a remark below (see equation (7.15).

Stress resultants and stress couples; equations of motion in spatial description

Associated with the motion $t \mapsto \Phi_t \in \mathcal{C}$ one assumes the existence of smooth vector fields $n_\alpha(X^0, t)$, $m_\alpha(X^0, t)$, $(\alpha = 1, 2)$, and $\rho_{ref}(X^0, t)$ interpreted as internal resultant force, internal resultant torque, and density per unit of area. These fields satisfy the following *spatial local form* of the equations of motion

$$\begin{aligned} n_{\alpha,\alpha} + \overline{n} &= \rho_{ref} h \dot{V} \ , \\ m_{\alpha,\alpha} + \varphi_{,\alpha} \times n_\alpha + \overline{m} &= \rho_{ref} K w \ , \end{aligned} \tag{7.10}$$

where $\overline{n}$, $\overline{m}$ are the applied body force and torque, and $\rho_{ref}h$, $\rho_{ref}K$ are intertia coefficients. Typically, for constant thickness plates, $\rho_{ref}K = \rho_{ref}h^3/12$. Note that $w \cdot t = 0$, that is $w \in T_I S_t$ for each $X^0 \in \Omega$ and $t \in [0, T]$. Equations (7.10) are the local statement in the spatial description of balance of linear momentum and balance of angular momentum, respectively. We refer to Libai and Simmonds [1983] for a derivation of these equations from the three dimensional theory.

Remark. From the point of view of the three-dimensional theory, the right-hand-side of (7.10) agrees with the standard definition of linear and angular momentum per unit of reference surface relative to the mid-plane. To see this, consider configurations $\tilde{\Phi} : \mathcal{B} \longrightarrow \mathbb{R}^3$ of the form $\tilde{\Phi} := \varphi + \xi \Lambda E$, where $\xi \in [-h/2, h/2]$. The angular momentum π relative to the mid-plane is then given by

$$\begin{aligned} \pi &:= \int_{-h/2}^{h/2} (\tilde{\Phi} - \varphi) \times \rho_{ref} \dot{\tilde{\varphi}} \, d\xi \\ &= \int_{-h/2}^{h/2} \rho_{ref} \xi \Lambda E \times (\dot{\phi} + \dot{\Lambda} E) \, d\xi \\ &= \frac{\rho_{ref} h^3}{12} \Lambda \left[E \times \Lambda^T \dot{\Lambda} E \right] \end{aligned}$$

$$= \frac{\rho_{ref} h^3}{12} \Lambda[E \times \hat{W} E] = \frac{\rho_{ref} h^3}{12} \Lambda[E \times (W \times E)]$$

$$= \frac{\rho_{ref} h^3}{12} \Lambda W = \frac{\rho_{ref} h^3}{12} w , \tag{7.11}$$

where we have used the fact that $\hat{W} \in T_I S_E$ so that $W \cdot E = 0$. ■

Convective equations of motion

An alternative form of the balance laws, which is particularly convenient in computational applications, is obtained from the convective description. As far as we are aware, this form of the equations has not been stated explicitly in the literature. As in the rod model discussed previously, the basic idea is to pull-back the spatial balance laws (7.10) with the orthogonal transformation $\Lambda : \Omega \longrightarrow S_E$. Accordingly, one defines vector fields $N(X^0, t)$, $M(X^0, t)$ by the expressions

$$N(X^0, t) := \Lambda^T n(X^0, t) , \qquad M(X^0, t) := \Lambda^T m(X^0, t) . \tag{7.12}$$

By making use of the relation

$$m_{\alpha,\beta} = \Lambda[M_{\alpha,\beta} + \Omega_\beta \times M_\alpha] , \tag{7.13}$$

and the analogous expression relating n_α and N_α, a straightforward calculation yields the following statements of linear and angular momentum:

$$\rho_{ref} h[\dot{\mathcal{V}} + W \times \mathcal{V}] = N_{\alpha,\alpha} + \Omega_\alpha \times N_\alpha + \bar{N} . \tag{7.14a}$$

$$\rho_{ref} K W = M_{\alpha,\alpha} + \Omega_\alpha \times M_\alpha + \Gamma_\alpha \times N_\alpha + \bar{M} . \tag{7.14b}$$

Remark. Within the context of the three dimensional theory, n_α and m_α represent resultant forces and resultant torques relative to the mid-surface of the distribution of stress acting on sections S_α. As in the rod model discussed above, γ_α, ω_α and Γ_α, Ω_α are uniquely determined in the sense that the stress power is given by

$$\int_{\Omega\times\left[-\frac{h}{2},\frac{h}{2}\right]} P : \dot{F}\, d\Omega\, d\xi \;=\; \int_{\Omega}\left[n_\alpha \cdot \overset{\nabla}{\gamma}_\alpha + m_\alpha \cdot \overset{\nabla}{\omega}_\alpha\right] d\Omega$$
$$= \int_{\Omega}\left[N_\alpha \cdot \dot{\Gamma}_\alpha + M_\alpha \cdot \dot{\Omega}_\alpha\right] d\Omega\ , \tag{7.15}$$

where P is the first Piola-Kirchoff stress tensor, F the deformation gradient of the configuration $\varphi := \varphi(X^0, t) + \xi t(X^0, t)$, and $(\cdot)^\nabla = (\cdot)^{\cdot} - w \times (\cdot)$ is a *co-rotated rate.* ■

Further reduction of the convective equations of motion

In contrast with the rod model considered in Section 5, the convective equations of motion (7.14) are amenable to further reduction. The reason for this additional reduction -- which can be carried out either in the spatial or in the convective descriptions -- is that only $S_E \subset SO(3)$ enters in the configuration space of the plate, not the entire $SO(3)$ as in the rod model.

Proceeding in the convected description, to carry out the reduction of equations (7.14) we first note that since $\Omega_\alpha, \hat{M}_\alpha \in T_I S_E$, it follows that $\Omega_\alpha \cdot E = M_\alpha \cdot E = 0$ and so $\Omega_\alpha \times M_\alpha$ is parallel to E. We exploit this fact by introducing the decomposition

$$N_\alpha =: N_\alpha^0 + Q_\alpha E\ , \qquad \Gamma_\alpha =: \Gamma_\alpha^0 + \Xi_\alpha E\ . \tag{7.16}$$

A straightforward calculation then yields

$$\Gamma_\alpha \times N_\alpha = \Gamma_\alpha^0 \times N_\alpha^0 + (\Gamma_\alpha^0 \times E)Q_\alpha - (N_\alpha^0 \times E)\Xi_\alpha\ ,$$
$$\Omega_\alpha \times N_\alpha = \Omega_\alpha \times N_\alpha^0 + (\Omega_\alpha \times E)Q_\alpha\ . \tag{7.17}$$

Making use of (7.16) and (7.17), balance of angular momentum (7.14b) now reduces to

$$\Omega_\alpha \times M_\alpha + \Gamma_\alpha^0 \times N_\alpha^0 = 0\ , \tag{7.18a}$$

$$M_{\alpha,\alpha} + \left(\Gamma_\alpha^0 \times E\right)Q_\alpha - \left(N_\alpha^0 \times E\right)\Xi_\alpha + \overline{M}^{\,0} = \rho_{ref} K W\ . \tag{7.18b}$$

The structure of equations (7.18) suggests the introduction of the following notation:

$$\begin{aligned} \Omega^0_\alpha &:= \Omega_\alpha \times E & \qquad & & \Omega_\alpha &= E \times \Omega^0_\alpha\,, \\ W^0 &:= W \times E & & \text{or} & W &= E \times W^0\,, \\ M^0_\alpha &:= M_\alpha \times E & & & M_\alpha &= E \times M^0_\alpha\,. \end{aligned} \tag{7.19}$$

In addition, we denote by $P_E := I - E \otimes E$ the orthogal projection parallel to E. Since $W \times \mathcal{V} = (\,E \times W^0\,) \times \mathcal{V}$, the convected acceleration may be expressed as

$$\Lambda^T \ddot{\phi} = \left[\dot{\mathcal{V}}^0 + \mathcal{v} W^0\right] + \left[\,\dot{\mathcal{v}} - W^0 \cdot \mathcal{V}^0\right] E\,, \tag{7.20a}$$

where

$$\mathcal{V}^0 := P_E \mathcal{V}\,, \quad \text{and} \quad \mathcal{v} := E \cdot \mathcal{V}\,. \tag{7.20b}$$

We then have the following five equations of balance of momentum in the convected representation:

$$\begin{aligned} N^0_{\alpha,\alpha} + \Omega^0_\alpha Q_\alpha + \overline{N}^{\,0} &= \rho_{ref} h\left[\dot{\mathcal{V}}^0 + \mathcal{v} W^0\right]\,, \\ Q_{\alpha,\alpha} - \Omega^0_\alpha \cdot N^0_\alpha + \overline{Q} &= \rho_{ref} h\left[\dot{\mathcal{v}} - W^0 \cdot \mathcal{V}^0\right]\,, \\ M^0_{\alpha,\alpha} - \Gamma^0_\alpha Q_\alpha + N^0_\alpha \Xi_\alpha + \overline{M}^{\,0} &= \rho_{ref} K W^0\,. \end{aligned} \tag{7.21}$$

Remark. *Interpretation of* W^0 *and* Ω^0_α. The equations of motion (7.21) along with conditions (7.18a) are in fact the convected form of the equations of motion corresponding to a *director description* of the plate, as in (7.6a,b). The vector field W^0 is the ***convected director velocity*** as the following identity shows:

$$\Lambda^T t = \Lambda^T \Lambda E = \hat{W} E \equiv W \times E =: W^0\,.$$

An entirely analogous interpretation holds for Ω^0_α. ■

Constitutive equations

In view of expression (7.15) for the stress power, one characterizes (isothermal) hyperelastic response by assuming the existence of a free energy function of the form

$$\Psi(X^0, \Gamma^0_\alpha, \Xi_\alpha, \Omega^0_\alpha).$$

As in the rod model considered in Section 5, this form of the free energy in the convective description arises naturally by postulating invariance under left action of the Euclidean group; i.e., material frame indifference. We then have the equations

$$N^0_\alpha = \frac{\partial\Psi\left(X^0, \Gamma^0_\alpha, \Xi_\alpha, \Omega^0_\alpha\right)}{\partial\Gamma^0_\alpha}, \tag{7.22a}$$

$$M^0_\alpha = \frac{\partial\Psi\left(X^0, \Gamma^0_\alpha, \Xi_\alpha, \Omega^0_\alpha\right)}{\partial\Omega^0_\alpha}, \tag{7.22b}$$

$$Q_\alpha = \frac{\partial\Psi\left(X^0, \Gamma^0_\alpha, \Xi_\alpha, \Omega^0_\alpha\right)}{\partial\Xi_\alpha}. \tag{7.22c}$$

We note that equation (7.18a), arising from balance of angular momentum about E can be expressed as

$$\Omega^0_\alpha \times M^0_\alpha + \Gamma^0_\alpha \times N^0_\alpha = 0.$$

From this condition one obtains the following restriction on the constitutive equations (7.22):

$$\Omega^0_\alpha \times \frac{\partial\Psi\left(X^0, \Gamma^0_\alpha, \Xi_\alpha, \Omega^0_\alpha\right)}{\partial\Omega^0_\alpha} + \Gamma^0_\alpha \times \frac{\partial\Psi\left(X^0, \Gamma^0_\alpha, \Xi_\alpha, \Omega^0_\alpha\right)}{\partial\Gamma^0_\alpha} = 0. \tag{7.23}$$

Condition (7.23) is analogous to the symmetry of the Cauchy and second Piola-Kirchoff stress tensor in three dimensional continuum mechanics, which in turn results from balance of angular momentum.

Hamiltonian in the convective picture

One defines the *convected linear and angular momenta* by the expressions

$$
\begin{aligned}
\mathcal{M} &= \mathcal{M}^0 + \mathcal{M}E := \rho_{ref}h[\mathcal{V}^0 + \mathcal{V}E] \\
\Pi &:= \rho_{ref}\mathbb{K}W \\
\Pi^0 &:= \Pi \times E \qquad \equiv \rho_{ref}\mathbb{K}W^0 \,.
\end{aligned}
\tag{7.24}
$$

Thus, in absence of body forces and assuming for simplicity homogeneous Dirichlet boundary conditions, the Hamiltonian is given by

$$
\begin{aligned}
H := \frac{1}{2}\int_\Omega \left\{ (\rho_{ref}h)^{-1}\left[\|\mathcal{M}^0\|^2 + \mathcal{M}^2\right] + (\rho_{ref}\mathbb{K})^{-1}\|\Pi^0\|^2 \right\} d\Omega \\
+ \int_\Omega \Psi\left(X^0, \Gamma^0_\alpha, \Xi_\alpha, \Omega^0_\alpha\right) d\Omega \,.
\end{aligned}
\tag{7.25}
$$

Convective Equations of Motion in Director Notation

For completeness we record below the field equations governing the proposed plate model in director notation. As in Section 7.2, let $\{E_I\}$ be the standard basis in $\mathbb{R}^3$, with $E_3 \equiv E$, $\{X_\alpha\}$ covering $\Omega \subset \mathbb{R}^2$, and $X_3 \equiv \xi \in [-h/2, h/2]$. Define

$$
\begin{aligned}
\mathbf{M}^0 &:= M^0_\alpha \otimes E_\alpha \,, &\quad \mathbf{N}^0 &:= N^0_\alpha \otimes E_\alpha \,, &\quad \mathbf{Q} &:= Q_\alpha E_\alpha \,, \\
\mathbf{W}^0 &:= \Omega^0_\alpha \otimes E_\alpha \,, &\quad \mathbf{G}^0 &:= \Gamma^0_\alpha \otimes E_\alpha \,, &\quad \mathbf{X} &:= \Xi_\alpha E_\alpha \,,
\end{aligned}
\tag{7.26}
$$

The momentum equations (7.21) may then be recast in the following form:

$$
\begin{aligned}
\mathcal{DIV}\mathbf{N}^0 + \mathbf{W}^0\mathbf{Q} + \overline{N}^0 &= \rho_{ref}h[\dot{\mathcal{V}}^0 + \mathcal{V}W^0] \,, \\
\mathcal{DIV}\mathbf{Q} - \mathbf{W}^0 : \mathbf{N}^0 + \overline{Q} &= \rho_{ref}h[\dot{\mathcal{V}} - W^0 \cdot \mathcal{V}^0] \,,
\end{aligned}
\tag{7.27}
$$

$$\mathcal{DIV}\,\mathfrak{M}^0 - \mathfrak{B}^0\mathfrak{Q} + \mathfrak{N}^0\mathfrak{X} + \overline{\mathrm{M}}^0 = \rho_{ref}\mathrm{K}\mathrm{W}^0 .$$

This director form of the equations is particularly convenient for the treatment of constitutive equations, as the following example illustrates.

Example A particular class of constitutive equations satisfying the invariance requirement (7.23) may be constructed by an argument that mimics that of the three dimensional theory. First, we assume the following more restrictive condition that implies (7.23):

$$\Omega_\alpha^0 \times \mathrm{M}_\alpha^0 = 0 , \quad \text{and} \quad \Gamma_\alpha^0 \times \mathrm{N}_\alpha^0 = 0 . \tag{7.28}$$

(a) Constitutive equations for N_α^0. Assume that N_α^0 depends only on Γ_α.

Define the following stress tensors:

$$\mathfrak{S}^0 := (\mathfrak{B}^0)^{-1}\,\mathfrak{N}^0 , \quad \mathfrak{T}^0 := \mathfrak{N}^0(\mathfrak{B}^0)^{\mathrm{T}} . \tag{7.29}$$

Then, condition $(7.28)_1$ is equivalent to the symmetry of $\mathfrak{S}^0$ and $\mathfrak{T}^0$; that is,

$$\Gamma_\alpha^0 \times \mathrm{N}_\alpha^0 = 0 \iff \mathfrak{S}^0 = (\mathfrak{S}^0)^{\mathrm{T}} \iff \mathfrak{T}^0 = (\mathfrak{T}^0)^{\mathrm{T}} \tag{7.30}$$

To complete the development, we define strain tensors conjugate to $\mathfrak{S}^0$ and $\mathfrak{T}^0$ by

$$\mathfrak{E}^0 := \frac{1}{2}\left[(\mathfrak{B}^0)^{\mathrm{T}}\mathfrak{B}^0 - 1\right] , \quad \mathfrak{H}^0 := \frac{1}{2}\left[1 - (\mathfrak{B}^0)^{-\mathrm{T}}\mathfrak{B}^{0\,-1}\right] \tag{7.31}$$

A direct calculation shows that these tensors satisfy the stress power relation

$$\mathrm{N}_\alpha^0 \cdot \dot{\Gamma}_\alpha^0 = \mathfrak{S}^0 : \dot{\mathfrak{E}}^0 = \mathfrak{T}^0 : \dot{\mathfrak{H}}^0 . \tag{7.32}$$

Accordingly, we define constitutive equations of the form

$$\mathfrak{S}^0 = \frac{\partial\overline{\Psi}(\mathrm{X}^0, \mathfrak{E}^0)}{\partial\mathfrak{E}^0} , \quad \mathfrak{T}^0 = \frac{\partial\overline{\Psi}(\mathrm{X}^0, \mathfrak{H}^0)}{\partial\mathfrak{H}^0} . \tag{7.33}$$

The net result is an uncoupled set of constitutive equations that satisfy at the outset restriction (7.23). In particular, one may consider a *linear relation* between $\mathbb{S}^0$ and $\mathbb{E}^0$.

(b) Constitutive equations for M^0_α. It is possible to construct an uncoupled equation for M^0_α of the form (7.22b) satisfying at the outset the restriction $(7.28)_1$. In particular, this relation can be linear.

§8. *The Hamiltonian Structure for the Geometrically Exact Plate Model*

In this section we develop the Hamiltonian structure for the plate model in the convective representation with governing equations summarized in the previous section . We show that these equations are Hamiltonian relative to a non-canonical Poisson structure in the space $\mathcal{P}$ of convected variables

$$\{(\Gamma^0{}_{\alpha}, \Xi_{\alpha}, \Omega^0{}_{\alpha}), \; (\mathcal{M}^0, \mathcal{M}, \Pi^0)\}.$$

The First Reduced Bracket

Our derivation of the corresponding Poisson bracket follows the same reduction scheme employed both in three dimendional elasticity and in our treatment of geometrically exact rods, and can be outlined as follows. We start with the canonical Hamiltonian structure on the cotangent bundle of the configuration space, $T^*\mathcal{C}$, and the corresponding canonical bracket. From this bracket one obtains by left reduction by SO(3); i.e., by material frame indiference, a Poisson bracket for the plate model, which is the counterpart of the Poisson bracket derived for the rod model. An important difference, however, is that one can further reduce this bracket by enforcing the condition that only a part of SO(3), namely $S_E \subset SO(3)$, enters in the configuration space. This additional reduction is the result of an extra symmetry of the Hamiltonian, which is now invariant with respect to rotations about the axis $\mathbf{t}$. Physically , this symmetry corresponds to the fact that no inertia and no stiffness is associated with rotations about the director $\mathbf{t}$.

The bracket in terms of the variables $\{(\Gamma_{\alpha}, \Omega_{\alpha}), (\mathcal{M}, \Pi)\}$ is derived from the canonical bracket in material variables in the same way as for the rod model. This bracket is as follows:

$$\begin{array}{c} \{\tilde{f}, \tilde{g}\} \\ \text{(canonical)} \end{array} \quad \left\{ \begin{array}{l} -\displaystyle\int_{\Omega} \Big\{ \Big(\frac{\partial \tilde{f}}{\partial \Gamma_{\alpha}}\Big)_{,\alpha} \cdot \frac{\partial \tilde{g}}{\partial \mathcal{M}} - \Big(\frac{\partial \tilde{g}}{\partial \Gamma_{\alpha}}\Big)_{,\alpha} \cdot \frac{\partial \tilde{f}}{\partial \mathcal{M}} \\[2ex] \qquad + \Big(\dfrac{\partial \tilde{f}}{\partial \Omega_{\alpha}}\Big)_{,\alpha} \cdot \dfrac{\delta \tilde{g}}{\partial \Pi} - \Big(\dfrac{\partial \tilde{g}}{\partial \Omega_{\alpha}}\Big)_{,\alpha} \cdot \dfrac{\partial \tilde{f}}{\partial \Pi} \Big\} \, d\Omega \end{array} \right. \tag{8.1a}$$

$$\text{(interaction)} \quad \Big\{ -\int_{\Omega} \Big\{ \Omega_{\alpha} \cdot \Big[\frac{\partial \tilde{f}}{\partial \Gamma_{\alpha}} \times \frac{\partial \tilde{g}}{\partial \mathcal{M}} - \frac{\partial \tilde{g}}{\partial \Gamma_{\alpha}} \times \frac{\partial \tilde{f}}{\partial \mathcal{M}} \Big] \Big\} \, d\Omega \tag{8.1b}$$

(Lie-Poisson for a semi-direct product)

$$\left\{ \begin{aligned} & - \int_\Omega \Big\{ \Omega_\alpha \cdot \Big[\frac{\partial \tilde f}{\partial \Omega_\alpha} \times \frac{\partial \tilde g}{\partial \Pi} - \frac{\partial \tilde g}{\partial \Omega_\alpha} \times \frac{\partial \tilde f}{\partial \Pi} \Big] \\ & \qquad + \Gamma_\alpha \cdot \Big[\frac{\partial \tilde f}{\partial \Gamma_\alpha} \times \frac{\partial \tilde g}{\partial \Pi} - \frac{\partial \tilde g}{\partial \Gamma_\alpha} \times \frac{\partial \tilde f}{\partial \Pi} \Big] \\ & \qquad + \mathcal{M} \cdot \Big[\frac{\partial \tilde f}{\partial \mathcal{M}} \times \frac{\partial \tilde g}{\partial \Pi} - \frac{\partial \tilde g}{\partial \mathcal{M}} \times \frac{\partial \tilde f}{\partial \Pi} \Big] \Big\} \, d\Omega \\ & - \int_\Omega \Pi \cdot \Big[\frac{\partial \tilde f}{\partial \Pi} \times \frac{\partial \tilde g}{\partial \Pi} \Big] \, d\Omega \end{aligned} \right. \tag{8.1c}$$

The Further Reduced Bracket

The further reduction is acomplished by using the change of variables in (7.19) given by

$$\Gamma_\alpha = \Gamma^0{}_\alpha + \Xi_\alpha E \qquad \mathcal{M} = \mathcal{M}^0 + \mathscr{M} E \tag{8.2a}$$

$$\Omega_\alpha = E \times \Omega^0{}_\alpha \qquad \Pi = E \times \Pi^0 \tag{8.2b}$$

Since $\hat{\Omega}_\alpha \in T_I S_E$ and $\hat{\Pi} \in T_I S_E$, we have the constraints

$$\Pi \cdot E = 0 \qquad \text{and} \qquad \Omega_\alpha \cdot E = 0, \tag{8.3}$$

and therefore

$$\Pi \cdot \Big(\frac{\partial \tilde f}{\partial \Pi} \times \frac{\partial \tilde g}{\partial \Pi} \Big) = 0 \tag{8.4}$$

and

$$\Omega_\alpha \cdot \Big(\frac{\partial \tilde f}{\partial \Omega_\alpha} \times \frac{\partial \tilde g}{\partial \Pi} \Big) = 0 \tag{8.5}$$

From (8.2a) and the chain rule, we get

$$\frac{\partial \tilde{f}}{\partial \Gamma_\alpha} = \frac{\partial f}{\partial \Gamma^0{}_\alpha} + E\,\frac{\partial f}{\partial \Xi_\alpha} \tag{8.6}$$

where $\tilde{f}$ is a function of $\{(\Gamma_\alpha, \Omega_\alpha), (\mathfrak{M}, \Pi)\}$, as in the bracket above, and f is a function of the variables $\{(\Gamma^0{}_\alpha, \Xi_\alpha, \Omega^0{}_\alpha), (\mathfrak{M}^0, \mathcal{M}, \Pi^0)\}$ which are related to the preceding variables through equations (8.2). Similarly one has

$$\frac{\partial g}{\partial \Pi^0} = \frac{\partial \tilde{g}}{\partial \Pi} \times E \tag{8.7}$$

and

$$\frac{\partial \tilde{f}}{\partial \mathfrak{M}} = \frac{\partial f}{\partial \mathfrak{M}^0} + E\,\frac{\partial f}{\partial \mathcal{M}} \tag{8.8}$$

Substitution of (8.4) - (8.8) into (8.1) gives the following bracket:

Theorem 8.1. *The reduced bracket on the space* $\mathcal{P}$ *of the variables*

$$\{(\Gamma^0{}_\alpha, \Xi_\alpha, \Omega^0{}_\alpha), (\mathfrak{M}^0, \mathcal{M}, \Pi^0)\}$$

is given by

$$\{f,g\} =$$

$$\text{(interaction)}\quad \left\{ \begin{aligned} &\int_\Omega \Big\{ \Omega^0{}_\alpha \cdot \Big[\frac{\partial f}{\partial \mathfrak{M}^0}\,\frac{\partial g}{\partial \Xi_\alpha} - \frac{\partial g}{\partial \mathfrak{M}^0}\,\frac{\partial f}{\partial \Xi_\alpha} \Big] \\ &\quad + \Omega^0{}_\alpha \cdot \Big[\frac{\partial f}{\partial \Gamma^0{}_\alpha}\,\frac{\partial g}{\partial \mathcal{M}} - \frac{\partial g}{\partial \Gamma^0{}_\alpha}\,\frac{\partial f}{\partial \mathcal{M}} \Big] \\ &\quad + \Gamma^0{}_\alpha \cdot \Big[\frac{\partial f}{\partial \Xi_\alpha}\,\frac{\partial g}{\partial \Pi^0} - \frac{\partial g}{\partial \Xi_\alpha}\,\frac{\partial f}{\partial \Pi^0} \Big] \end{aligned} \right.$$

$$+ \Xi_\alpha \cdot \left[\frac{\partial f}{\partial \Pi^0} \cdot \frac{\partial g}{\partial \Gamma^0_\alpha} - \frac{\partial g}{\partial \Pi^0} \cdot \frac{\partial f}{\partial \Gamma^0_\alpha} \right]\Big\} \, d\Omega$$

(Lie Poisson for a semidirect product)

$$+ \int_\Omega \Big\{ \mathcal{M}^0 \cdot \left[\frac{\partial f}{\partial \mathcal{M}} \frac{\partial g}{\partial \Pi^0} - \frac{\partial g}{\partial \mathcal{M}^0} \frac{\partial f}{\partial \Pi^0} \right]$$

$$+ \mathcal{M} \cdot \left[\frac{\partial f}{\partial \Pi^0} \cdot \frac{\partial g}{\partial \mathcal{M}^0} - \frac{\partial g}{\partial \Pi^0} \cdot \frac{\partial f}{\partial \mathcal{M}^0} \right]\Big\} \, d\Omega$$

(canonical)

$$- \int_\Omega \Big\{ \left(\frac{\partial f}{\partial \Omega^0_\alpha} \right)_{,\alpha} \cdot \frac{\partial g}{\partial \Pi^0} - \left(\frac{\partial g}{\partial \Omega^0_\alpha} \right)_{,\alpha} \cdot \frac{\partial f}{\partial \Pi^0}$$

$$+ \left(\frac{\partial f}{\partial \Gamma^0_\alpha} \right)_{,\alpha} \cdot \frac{\partial g}{\partial \mathcal{M}^0} - \left(\frac{\partial g}{\partial \Gamma^0_\alpha} \right)_{,\alpha} \cdot \frac{\partial f}{\partial \mathcal{M}^0}$$

$$+ \left(\frac{\partial f}{\partial \Xi_\alpha} \right)_{,\alpha} \cdot \frac{\partial g}{\partial \mathcal{M}} - \left(\frac{\partial g}{\partial \Xi_\alpha} \right)_{,\alpha} \cdot \frac{\partial f}{\partial \mathcal{M}} \Big\} \, d\Omega \tag{8.9}$$

As for the rod model discussed in Sections 5 and 6, from the expression (7.25) for the Hamiltonian and the bracket (8.9), we have the following:

Corollary 8.2. *Hamilton's equations in the form* $\dot{f} = \{ f, H \}$, *where* f *is an arbitrary function on the phase space given by theorem 8.1, with the bracket given by equation (8.9) and Hamiltonian given by equation (7.25) are equivalent to the following convective equations of motion:*

$$\dot{\mathcal{M}} = Q_{\alpha,\alpha} - \Omega^0_\alpha \cdot N^0_\alpha + W^0 \cdot \mathcal{M}^0$$

$$\dot{\mathcal{M}}^0 = N^0_{\alpha,\alpha} + \Omega^0_\alpha Q_\alpha - \mathcal{M} W^0$$

$$\dot{\Pi}^0 = \mathcal{M}^0_{\alpha,\alpha} + \Xi_\alpha N^0_\alpha - \Gamma^0_\alpha Q_\alpha$$

$$\dot{\Xi}_\alpha = \mathcal{V}_\alpha + W^0 \cdot \Gamma_\alpha - \Omega^0{}_\alpha \cdot \mathcal{V}^0 \qquad (8.10)$$

$$\dot{\Gamma}^0{}_\alpha = \mathcal{V}^0{}_{,\alpha} - W^0 \Xi_\alpha + \Omega^0{}_\alpha \mathcal{V}$$

$$\dot{\Omega}^0{}_\alpha = W^0{}_{,\alpha}$$

where we have set $N^0{}_\alpha = \partial\psi/\partial\Gamma^0{}_\alpha$, $Q_\alpha = \partial\psi/\partial\Xi_\alpha$ *and* $M^0{}_\alpha = \partial\psi/\partial\Omega^0{}_\alpha$.

The first three of these equations of motion have been derived already (see 7.21). The second group of three equations can be directly checked using the kinematic relations given in section 7.The corollary then follows by the general principles of reduction, but may also be verified by a direct computation.

Conclusions

In this paper we have presented a systematic development of the Hamiltonian structure for geometrically exact nonlinear elasticity, including solids, rods, and plates. We have emphasised the convective representation since it is numerically convenient and it is useful for the coupling with rigid body dynamics to obtain models for rigid bodies with flexible attatchments. In particular, our formulation of the geometrically exact plate model in the convected representation constitutes the natural counterpart of the classical Kirchhoff-Love and Reissner-Antman models for rods. The derivation of the Poisson structure follows the same lines as earlier works of Marsden, Ratiu, Weinstein and their coworkers, namely it is obtained by reduction from canonical brackets in the material (or Lagrangian) representation by reduction; ie, by the elimination of rotational symmetry (material frame indifference), and by introducing the Cauchy-Green tensor as a dynamical variable. This approach is consistent with the covariance investigations of nonlinear elasticity by Hughes, Marsden, and Simo. In a future work we shall use the Hamiltonian structures for geometrically exact rods and plates to study the nonlinear stability of rigid bodies with flexible attatchments, following the ideas of Krishnaprasad and Marsden.

It is clear from the literature and related work that these methods are much more significant and widely applicable than the results given here may directly indicate. For example, the attention to the proper geometry and the nonlinear context that is typified by the present investigation, is of benefit for numerical work, as has been demonstrated by Simo and VuQuoc. Also, other models

can be investigated; for example, it is clear from the literature (Iwinski and Turski, Marsden and Weinstein) that one can also include electromagnetic effects into the same formalism.

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